Theory of Stellar Oscillations

Margarida S. Cunha

COURSE 7

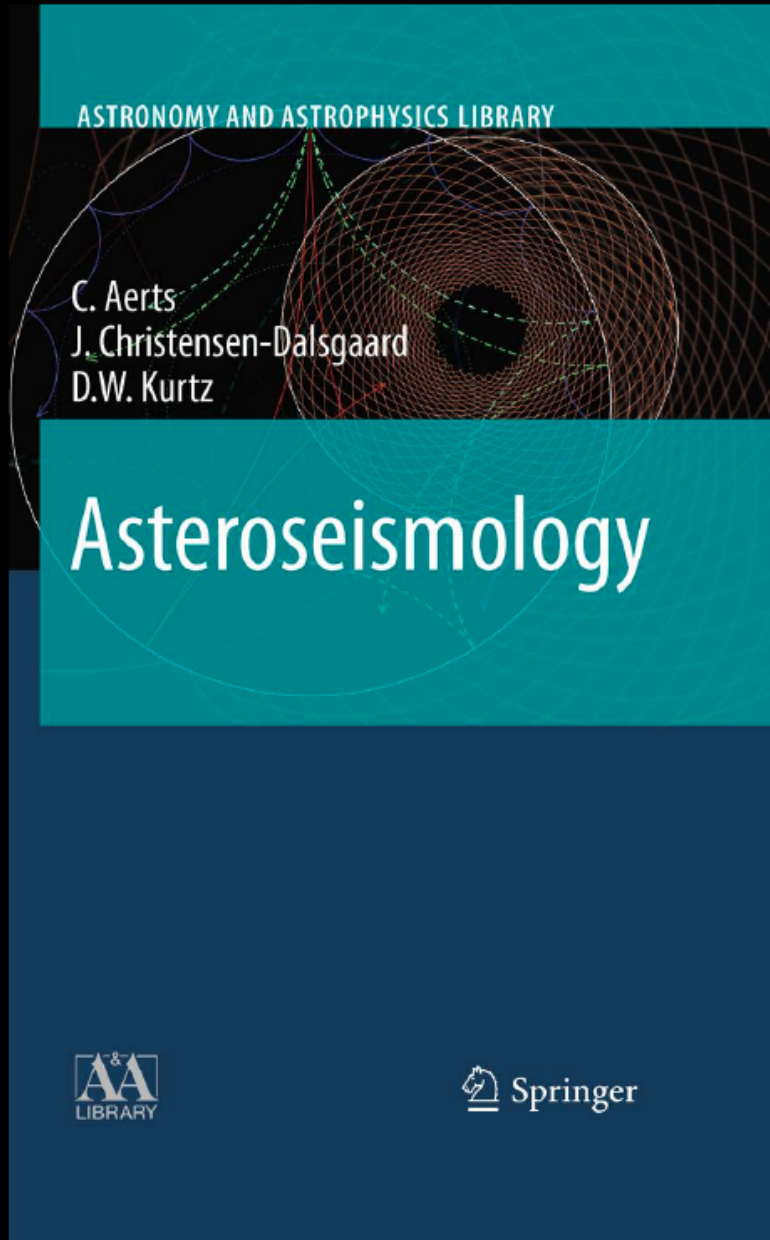
LINEAR ADIABATIC STELLAR PULSATION

Douglas O. Gough

Institute of Astronomy and
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, CB3 0HA, UK
and
Joint Institute for Laboratory Astrophysics,
University of Colorado, Boulder CO 80309, USA

J.-P. Zahn and J. Zinn-Justin, eds.
Les Houches, Session XLVII, 1987
Dynamique des fluides astrophysiques
Astrophysical fluid dynamics

399



Brief introduction

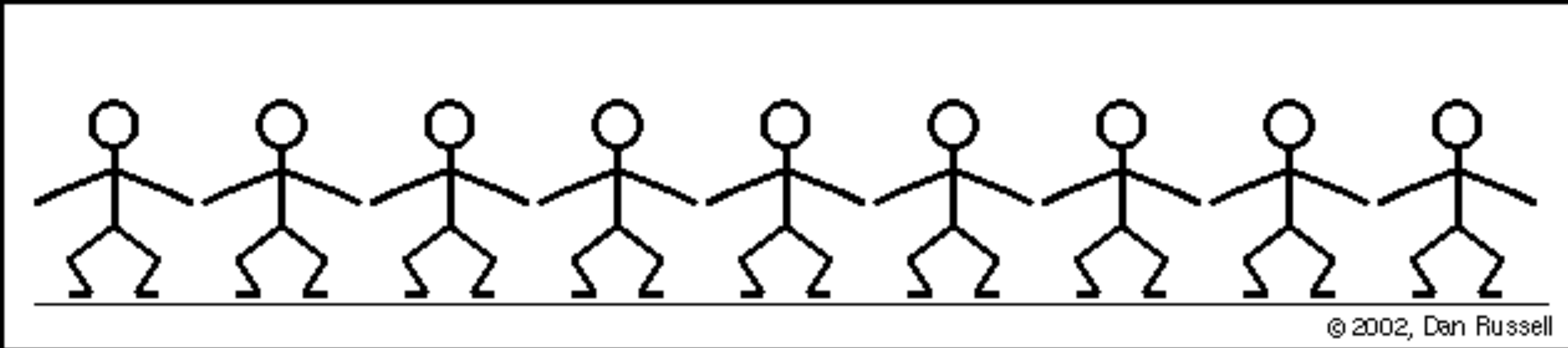
Asteroseismology How does it work?

How would you describe a wave?

Asteroseismology How does it work?

Wave: propagation of information (a perturbation) in space and time

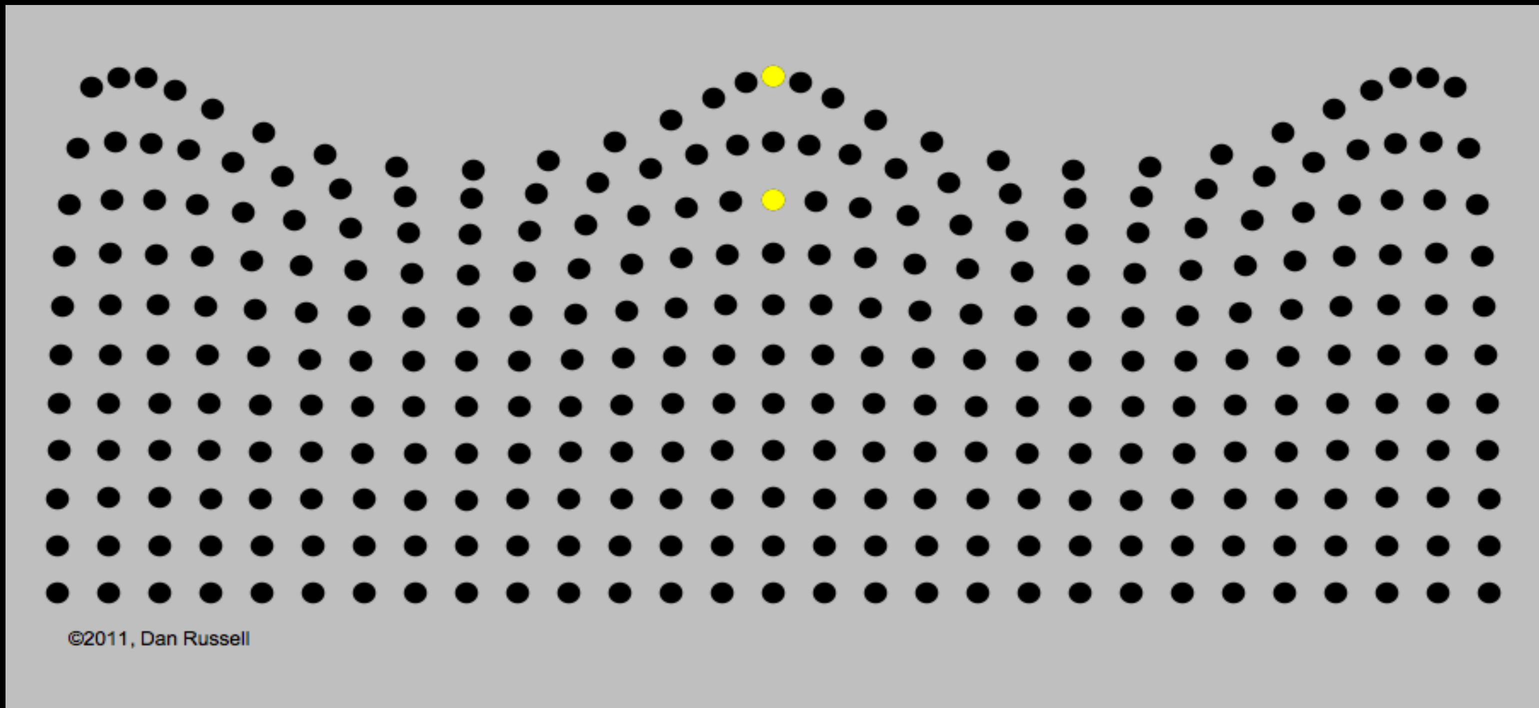
Wave in a supporting medium: material does not need to move from one point of the space to the other to propagate the information



Asteroseismology How does it work?

Wave: propagation of information (a perturbation) in space and time

Wave in a supporting medium: material does not need to move from one point of the space to the other to propagate the information



Asteroseismology How does it work?

Waves propagate within stars

Asteroseismology How does it work?

Waves propagate within stars

⇓

Wave properties (e.g. frequencies) depend on properties of the medium where they propagate (density, pressure, etc.)

Asteroseismology How does it work?

Waves propagate within stars

⇓

Wave properties (e.g. frequencies) depend on properties of the medium where they propagate (density, pressure, etc.)

⇓

Properties = f (interior)

Asteroseismology How does it work?

Waves propagate within stars

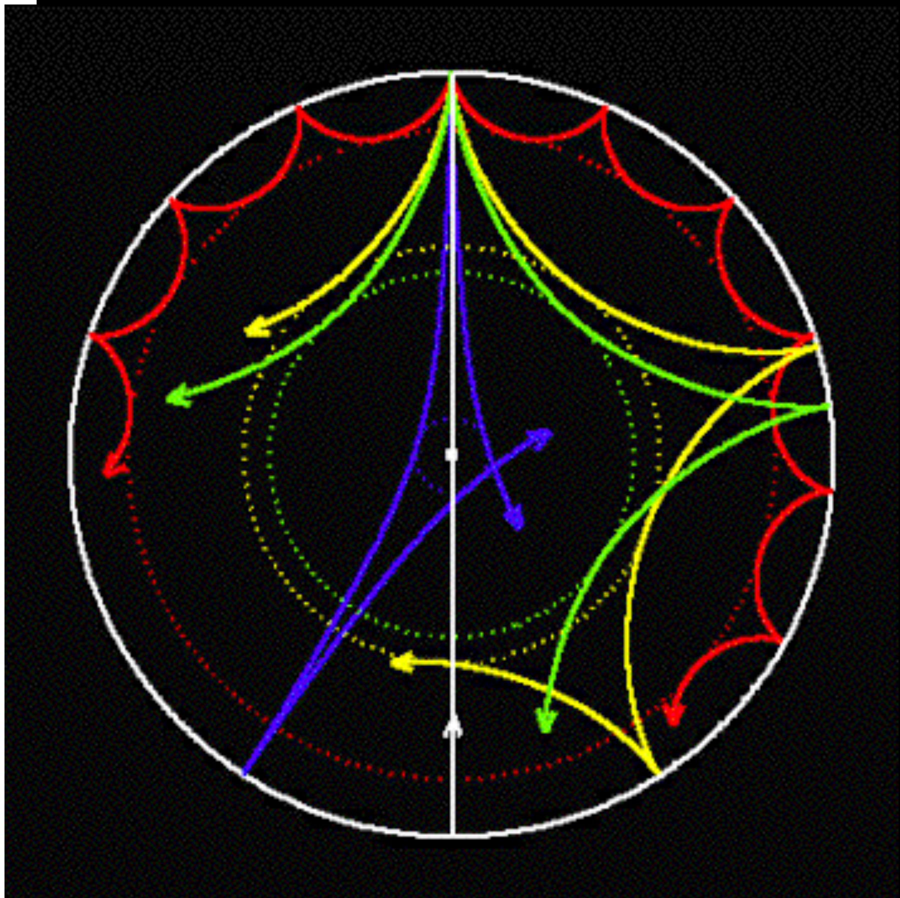
⇓

Wave properties (e.g. frequencies) depend on properties of the medium where they propagate (density, pressure, etc.)

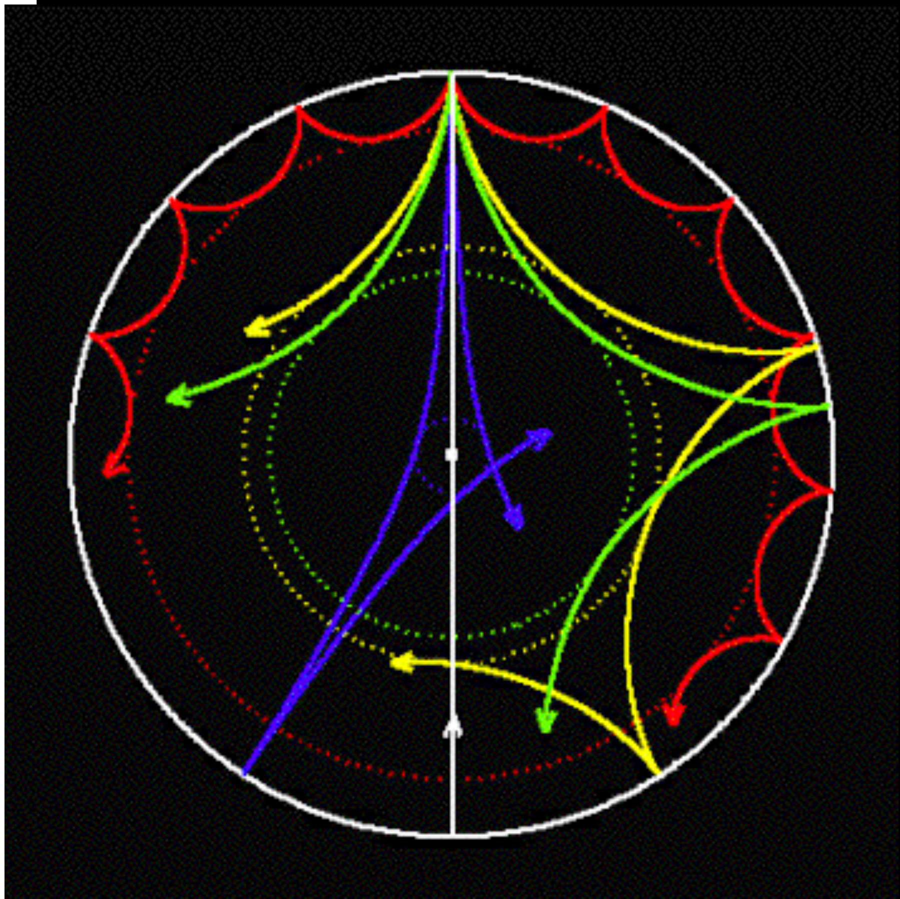
⇓

$$\text{Properties} = f\,(\text{interior})$$

Asteroseismology How does it work?



Asteroseismology How does it work?



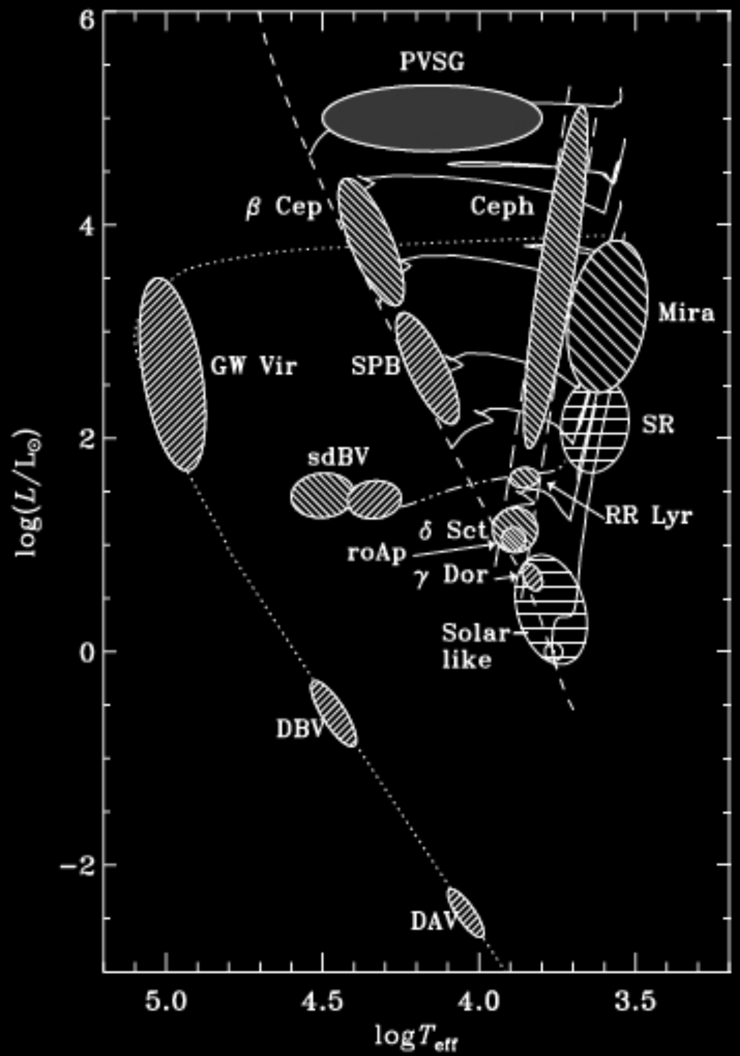
One mode ⇔ one piece of information

- Average information on propagation cavity

- With several modes one can hope to get localized information

Asteroseismology: Across the HR diagram

Kurtz 2010 adapted from Aerts et al. 2010

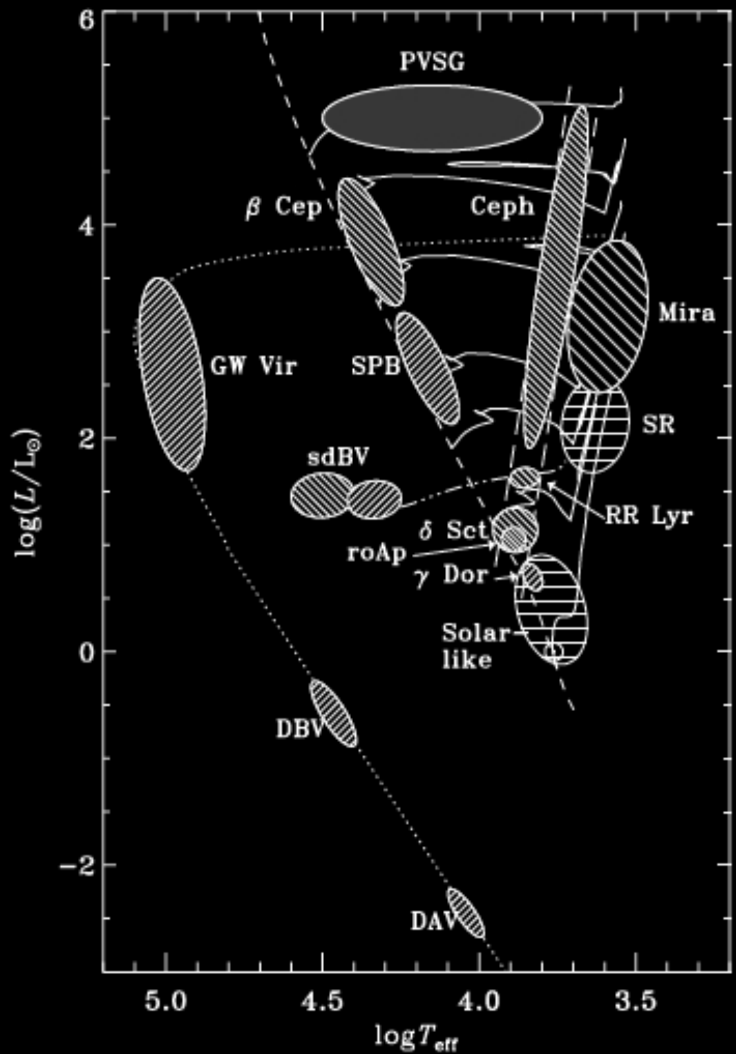


Asteroseismology: Classification

Origin
- Intrinsically unstable
  - Classical
- Intrinsically stable
  - Solar-like

Nature
- Acoustic waves
  - p modes
- Internal Gravity waves
  - g modes

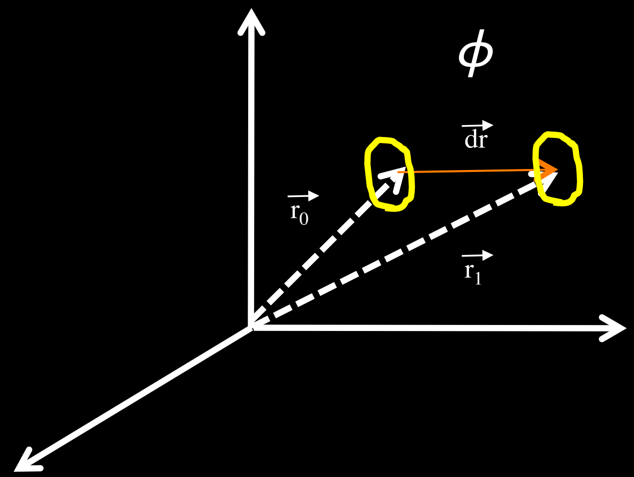
Kurtz 2010 adapted from Aerts et al. 2010



Hydrodynamics

Hydrodynamics

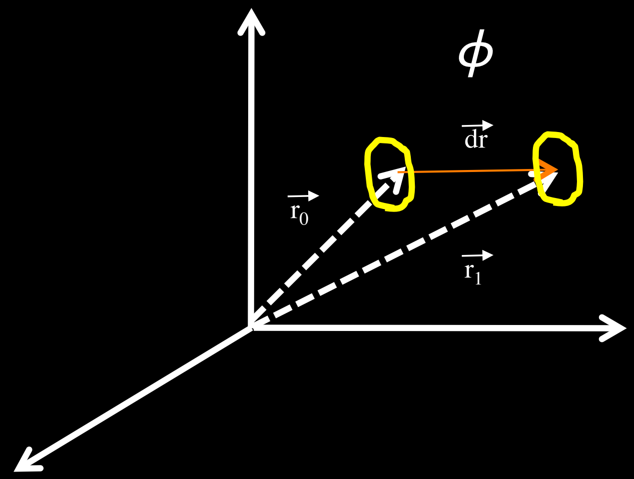
Assume that the gas can be treated as a continuum; Thermodynamic properties well defined at each position $\vec{r}$



Let ϕ be a scalar property of the gas.

Hydrodynamics

Assume that the gas can be treated as a continuum; Thermodynamic properties well defined at each position $\vec{r}$



Let ϕ be a scalar property of the gas.

Two ways to look at time evolution of ϕ:

1. At fixed position => Eulerian description
2. Following the motion => Lagrangian description

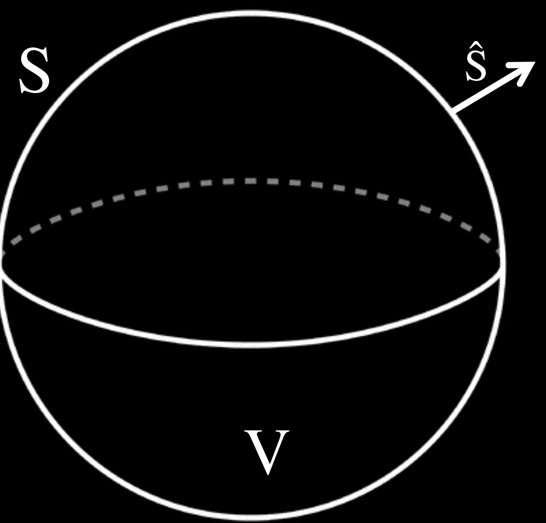
$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \nabla\phi \cdot \frac{D\vec{r}}{dt}$$
$$= \frac{\partial \phi}{\partial t} + \vec{v} \cdot \nabla\phi$$

Hydrodynamics

Continuity equation : The mass variation within a given volume V must equal, with opposite sign, the mass crossing the surface S that encloses the volume V.

Hydrodynamics

Continuity equation : The mass variation within a given volume V must equal, with opposite sign, the mass crossing the surface S that encloses the volume V.



$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{v})$$

ρ - density $\vec{v}$ - velocity

Hydrodynamics

Continuity equation
(conservation of mass)

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{v})$$

ρ - density $\vec{v}$ - velocity

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0$$

ρ - density $\vec{v}$ - velocity

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla\cdot\vec{v} \Leftrightarrow \frac{1}{V}\frac{DV}{Dt} = \nabla\cdot\vec{v}$$

ρ - density $\quad \vec{v}$ - velocity $\quad V$ - volume

Rate of expansion of the fluid

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla\cdot\vec{v} \Leftrightarrow \frac{1}{V}\frac{DV}{Dt} = \nabla\cdot\vec{v}$$

ρ - density $\vec{v}$ - velocity V - volume

Rate of expansion of the fluid

=> Acoustic waves require div $\vec{v} \neq 0$

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot\,\vec{v} = 0$$

ρ - density $\quad$ $\vec{v}$ - velocity

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0$$

ρ - density $\vec{v}$ - velocity

Equation of motion: The change in linear momentum of an element of fluid must equal the force acting on it by its surroundings.

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot\vec{v} = 0$$

ρ - density $\quad$ $\vec{v}$ - velocity

Equation of motion: The change in linear momentum of an element of fluid must equal the force acting on it by its surroundings.

$$\rho\frac{D\vec{v}}{Dt} = -\nabla p + \rho\vec{g} + \vec{F}$$

p - pressure $\quad$ $\vec{g} = -\nabla\phi$ - acceleration of gravity $\quad$ $\vec{F}$ -other body forces

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0$$

ρ - density $\vec{v}$ - velocity

Equation of motion (inviscid fluid)
(conservation of linear momentum)

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \rho \vec{g} + \vec{F}$$

p - pressure $\vec{g} = -\nabla \phi$ - acceleration of gravity $\vec{F}$ -other body forces per unit volume

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0$$

ρ - density $\vec{v}$ - velocity

Equation of motion (inviscid fluid)
(conservation of linear momentum)

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \rho \vec{g} + \vec{F}$$

p - pressure $\vec{g} = -\nabla \phi$ - acceleration of gravity $\vec{F}$ -other body forces per unit volume

+ Poisson equation $\nabla^2 \phi = 4\pi G \rho$

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot\vec{v} = 0$$

ρ - density $\vec{v}$ - velocity

Equation of motion (inviscid fluid)
(conservation of linear momentum)

$$\rho\frac{D\vec{v}}{Dt} = -\nabla p + \rho\vec{g} + \cancel{\vec{F}}$$

p - pressure $\vec{g} = -\nabla\phi$ - acceleration of gravity $\vec{F}$ -other body forces per unit volume

+ Poisson equation $\nabla^2\phi = 4\pi G\rho$

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\, \nabla\cdot\, \vec{\mathrm{v}} = 0$$

ρ - density $\qquad$ $\vec{\mathrm{v}}$ - velocity

Equation of motion (inviscid fluid)
(conservation of linear momentum)

$$\rho \frac{\mathrm{D}\vec{\mathrm{v}}}{\mathrm{D}t} = -\nabla p - \rho \nabla \phi$$

$$\nabla^2 \phi = 4\pi G \rho$$

p - pressure $\qquad$ ϕ - Gravitational potential

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\nabla\cdot\vec{\mathrm{v}} = 0$$

ρ - density $\vec{\mathrm{v}}$ - velocity

Equation of motion (inviscid fluid)
(conservation of linear momentum)

$$\rho\frac{\mathrm{D}\vec{\mathrm{v}}}{\mathrm{D}t} = -\nabla p - \rho\nabla\phi$$

$$\nabla^2\phi = 4\pi G\rho$$

p - pressure ϕ - Gravitational potential

Energy equation (first law of thermodynamics): the change in the internal energy of a system equals the heat supplied to the system minus the work done by the system on its surroundings.

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0$$

ρ - density $\vec{v}$ - velocity

Equation of motion (inviscid fluid)
(conservation of linear momentum)

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p - \rho \nabla \phi$$

$$\nabla^2 \phi = 4\pi G \rho$$

p - pressure ϕ - Gravitational potential

Energy equation (first law of thermodynamics): the change in the internal energy of a system equals the heat supplied to the system minus the work done by the system on its surroundings.

$$\frac{Dq}{Dt} = \frac{DE}{Dt} + p\frac{D(1/\rho)}{Dt}$$

q -heat supplied /mass E -internal energy /mass

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot\vec{v} = 0$$

ρ - density $\vec{v}$ - velocity

Equation of motion (inviscid fluid)
(conservation of linear momentum)

$$\rho\frac{D\vec{v}}{Dt} = -\nabla p - \rho\nabla\phi$$

$$\nabla^2\phi = 4\pi G\rho$$

p - pressure ϕ - Gravitational potential

Energy equation
(conservation of energy)

$$\frac{Dq}{Dt} = \frac{DE}{Dt} + p\frac{D(1/\rho)}{Dt}$$

q -heat supplied /mass E -internal energy /mass

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \vec{\mathrm{v}} = 0$$

ρ - density $\vec{\mathrm{v}}$ - velocity

Equation of motion (inviscid fluid)
(conservation of linear momentum)

$$\rho \frac{\mathrm{D}\vec{\mathrm{v}}}{\mathrm{D}t} = -\nabla p - \rho \nabla \phi$$

$$\nabla^2 \phi = 4\pi G \rho$$

p - pressure ϕ - Gravitational potential

Energy equation
(conservation of energy)

q -heat supplied /mass E -internal energy /mass

$\Gamma_1;\Gamma_3$ - adiabatic exponents

$$\frac{\mathrm{D}q}{\mathrm{D}t} = \frac{\mathrm{D}E}{\mathrm{D}t} + p\frac{\mathrm{D}(1/\rho)}{\mathrm{D}t} = \frac{1}{\rho(\Gamma_3 - 1)}\left(\frac{\mathrm{D}p}{\mathrm{D}t} - \frac{\Gamma_1 p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right)$$

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot\vec{v} = 0$$

ρ - density $\vec{v}$ - velocity

Equation of motion (inviscid fluid)
(conservation of linear momentum)

$$\rho\frac{D\vec{v}}{Dt} = -\nabla p - \rho\nabla\phi$$

$$\nabla^2\phi = 4\pi G\rho$$

p - pressure ϕ - Gravitational potential

Energy equation
(conservation of energy)

$$\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_{ad}$$

$$\frac{Dq}{Dt} = \frac{DE}{Dt} + p\frac{D(1/\rho)}{Dt} = \frac{1}{\rho(\Gamma_3 - 1)}\left(\frac{Dp}{Dt} - \frac{\Gamma_1 p}{\rho}\frac{D\rho}{Dt}\right)$$

q -heat supplied /mass E -internal energy /mass

$\Gamma_1;\Gamma_3$ - adiabatic exponents

Hydrodynamics

Following the fluid - Lagrangian description

Continuity equation
(conservation of mass)

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\nabla\cdot\vec{\mathrm{v}} = 0$$

ρ - density $\vec{\mathrm{v}}$ - velocity

Equation of motion (inviscid fluid)
(conservation of linear momentum)

$$\rho\frac{\mathrm{D}\vec{\mathrm{v}}}{\mathrm{D}t} = -\nabla p - \rho\nabla\phi$$

$$\nabla^2\phi = 4\pi G\rho$$

p - pressure ϕ - Gravitational potential

Energy equation
(conservation of energy)

$$\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_{ad}$$

$$\frac{\mathrm{D}q}{\mathrm{D}t} = \frac{\mathrm{D}E}{\mathrm{D}t} + p\frac{\mathrm{D}(1/\rho)}{\mathrm{D}t} = \\ = \frac{1}{\rho(\Gamma_3 - 1)}\left(\frac{\mathrm{D}p}{\mathrm{D}t} - \frac{\Gamma_1 p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right)$$

q -heat supplied /mass E -internal energy /mass

$\Gamma_1;\Gamma_3$ - adiabatic exponents

+ Equation of state

Perturbations

Perturbations

Equilibrium state:

- In static equilibrium
- Spherically symmetric

Perturbations

Equilibrium state:

- In static equilibrium
- Spherically symmetric

Small perturbations about equilibrium:

$f = f_0 + f'$ - *where* f' *is the Eulerian perturbation*

δf - *is the Lagrangian perturbation*

Perturbations

Equilibrium state:

- In static equilibrium
- Spherically symmetric

Small perturbations about equilibrium:

$f = f_0 + f'$ - where f' is the Eulerian perturbation
δf - is the Lagrangian perturbation

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \vec{v}) = 0$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla p' - \rho_0 \nabla \phi' - \rho' \nabla \phi_0$$

$$\nabla^2 \phi' = 4\pi G \rho'$$

$$\frac{\partial \delta q}{\partial t} = \frac{1}{\rho(\Gamma_{3,0} - 1)} \left(\frac{\partial \delta p}{\partial t} - \frac{\Gamma_{1,0} p_0}{\rho_0} \frac{\partial \delta \rho}{\partial t} \right)$$

Perturbations

Equilibrium state:

- In static equilibrium
- Spherically symmetric

Small perturbations about equilibrium:

$f = f_0 + f'$ - where f' is the Eulerian perturbation
δf - is the Lagrangian perturbation

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \vec{v}) = 0$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla p' - \rho_0 \nabla \phi' - \rho' \nabla \phi_0$$

$$\nabla^2 \phi' = 4\pi G \rho'$$

$$\cancel{\frac{\partial \delta q}{\partial t}} = \frac{1}{\rho(\Gamma_{3,0} - 1)} \left(\frac{\partial \delta p}{\partial t} - \frac{\Gamma_{1,0} p_0}{\rho_0} \frac{\partial \delta \rho}{\partial t} \right)$$

Adiabatic approximation

Perturbations

Equilibrium state:

- In static equilibrium
- Spherically symmetric

➡

Small perturbations about equilibrium:

$f = f_0 + f'$ - where f' is the Eulerian perturbation
δf - is the Lagrangian perturbation

$$\frac{\partial \rho'}{\partial t} + \nabla\cdot(\rho_0 \vec{\mathrm{v}}) = 0$$

$$\rho_0 \frac{\partial \vec{\mathrm{v}}}{\partial t} = -\nabla p' - \rho_0 \nabla \phi' - \rho' \nabla \phi_0$$

$$\nabla^2 \phi' = 4\pi G \rho'$$

$$\cancel{\frac{\partial \delta q}{\partial t}} = \frac{1}{\rho(\Gamma_{3,0}-1)}\left(\frac{\partial \delta p}{\partial t} - \frac{\Gamma_{1,0} p_0}{\rho_0}\frac{\partial \delta \rho}{\partial t}\right)$$

Adiabatic approximation

Characteristic time scale for radiation:
Sun as a whole: 10^7 years
Solar convection zone: 10^3 years

Perturbations

Equilibrium state:

- In static equilibrium
- Spherically symmetric

Small perturbations about equilibrium:

$f = f_0 + f'$ - where f' is the Eulerian perturbation
δf - is the Lagrangian perturbation

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \vec{v}) = 0$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla p' - \rho_0 \nabla \phi' - \rho' \nabla \phi_0$$

$$\nabla^2 \phi' = 4\pi G \rho'$$

$$\frac{\partial \delta p}{\partial t} = \frac{\Gamma_{1,0} p_0}{\rho_0} \frac{\partial \delta \rho}{\partial t}$$

Adiabatic approximation

Characteristic time scale for radiation:
Sun as a whole: 10^7 years
Solar convection zone: 10^3 years

Perturbations

Equilibrium state:

- In static equilibrium
- Spherically symmetric

Small perturbations about equilibrium:

$f = f_0 + f'$ - where f' is the Eulerian perturbation
δf - is the Lagrangian perturbation

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \vec{v}) = 0$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla p' - \rho_0 \nabla \phi' - \rho' \nabla \phi_0$$

$$\nabla^2 \phi' = 4\pi G \rho'$$

$$\frac{\partial \delta p}{\partial t} = \frac{\Gamma_{1,0} p_0}{\rho_0} \frac{\partial \delta \rho}{\partial t}$$

$$\vec{v} = \frac{\partial \delta \vec{r}}{\partial t}$$

Perturbations

Equilibrium state:

- In static equilibrium
- Spherically symmetric

Small perturbations about equilibrium:

$f = f_0 + f'$ - where f' is the Eulerian perturbation
δf - is the Lagrangian perturbation

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \vec{v}) = 0$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla p' - \rho_0 \nabla \phi' - \rho' \nabla \phi_0$$

$$\nabla^2 \phi' = 4\pi G \rho'$$

$$\frac{\partial \delta p}{\partial t} = \frac{\Gamma_{1,0} p_0}{\rho_0} \frac{\partial \delta \rho}{\partial t}$$

$$\vec{v} = \frac{\partial \delta \vec{r}}{\partial t}$$

$$\delta f = f' + \delta \vec{r} . \nabla f_0$$

Summary of perturbed equations

Linear adiabatic pulsation about a static, spherically symmetric equilibrium

$$\rho' + \nabla\cdot(\rho_0 \delta\vec{r}) = 0$$

$$\rho_0 \frac{\partial^2 \delta\vec{r}}{\partial t^2} = -\nabla p' - \rho_0 \nabla \phi' - \rho' \nabla \phi_0$$

$$\nabla^2 \phi' = 4\pi G \rho'$$

$$p' + \delta\vec{r}\cdot\nabla p_0 = \frac{\Gamma_{1,0}\, p_0}{\rho_0}(\rho' + \delta\vec{r}\cdot\nabla\rho_0)$$

Summary of perturbed equations

Linear adiabatic pulsation about a static, spherically symmetric equilibrium

$$\rho' + \nabla\cdot(\rho_0 \delta\vec{r}) = 0$$

$$\rho_0 \frac{\partial^2 \delta\vec{r}}{\partial t^2} = -\nabla p' - \rho_0 \nabla \phi' - \rho' \nabla \phi_0$$

$$\nabla^2 \phi' = 4\pi G \rho'$$

$$p' + \delta\vec{r}\cdot\nabla p_0 = \frac{\Gamma_{1,0} p_0}{\rho_0}(\rho' + \delta\vec{r}\cdot\nabla\rho_0)$$

Variables: 4 (ϱ', p', ϕ', $\delta\vec{r}$)

Equations: 4

Thus: system of equation is closed, so far as equilibrium quantities are known.

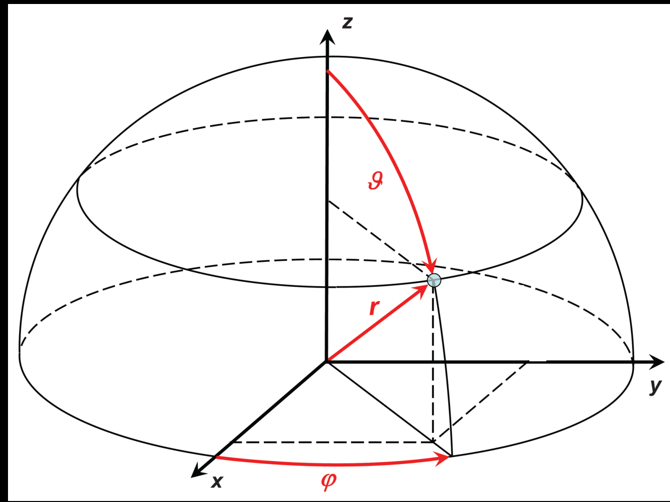
=> can solve it to get solutions for the 4 variables.

Solutions on a sphere

Solutions on a sphere

Consider the spherical coordinates (r,θ,φ)

Variables $(\rho', p', \phi', \delta\vec{r})$ are function of: r, θ, φ, t

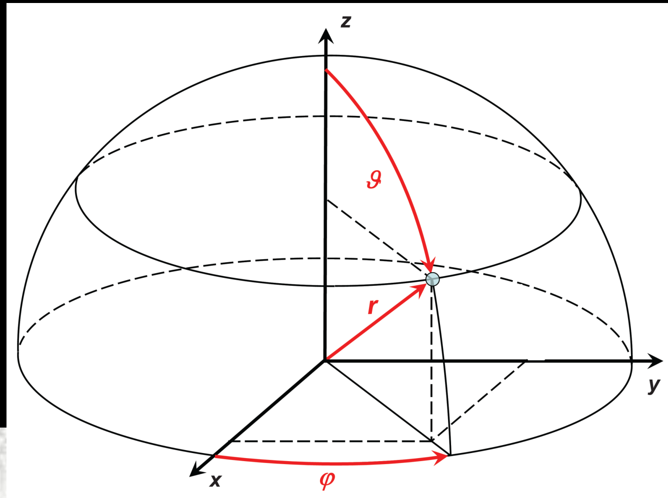


Solutions on a sphere

Consider the spherical coordinates (r,θ,φ)

Variables (ϱ', p', ϕ', $\delta\vec{r}$) are function of: r, θ, φ, t

The equations admit solutions of the type:



$$p'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[p'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\rho'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[\rho'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\phi'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[\phi'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\delta\vec{\mathrm{r}}(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}\left\{\left[\xi_r(r)Y_l^m\hat{\mathrm{a}}_\mathrm{r} + \xi_h(r)\left(\frac{\partial Y_l^m}{\partial\theta}\hat{\mathrm{a}}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\hat{\mathrm{a}}_\phi\right)\right]\mathrm{e}^{-i\omega t}\right\}$$

Solutions on a sphere

Consider the spherical coordinates (r,θ,φ)

Variables $(\varrho', p', \phi', \delta\vec{r})$ are function of: r, θ, φ, t

The equations admit solutions of the type:

$$p'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[p'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\rho'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[\rho'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\phi'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[\phi'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\delta\vec{\mathrm{r}}(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}\left\{\left[\xi_r(r)Y_l^m\hat{\mathrm{a}}_\mathrm{r} + \xi_h(r)\left(\frac{\partial Y_l^m}{\partial\theta}\hat{\mathrm{a}}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\hat{\mathrm{a}}_\phi\right)\right]\mathrm{e}^{-i\omega t}\right\}$$

Solutions on a sphere

Consider the spherical coordinates (r,θ,φ)

Variables (ϱ', p', ϕ', $\delta\vec{r}$) are function of: r, θ, φ, t

The equations admit solutions of the type:

$$p'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[p'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\rho'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[\rho'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\phi'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[\phi'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\delta\vec{\mathrm{r}}(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}\left\{\left[\xi_r(r)Y_l^m\hat{\mathrm{a}}_\mathrm{r} + \xi_h(r)\left(\frac{\partial Y_l^m}{\partial\theta}\hat{\mathrm{a}}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\hat{\mathrm{a}}_\phi\right)\right]\mathrm{e}^{-i\omega t}\right\}$$

Spherical Harmonics Y_l^m

l – *angular degree: the number of nodes on the sphere*

$$k_h = \frac{\sqrt{l(l+1)}}{R}$$

m - azimuthal order: $|m|$ =number of nodes along the equator

=> orientation on the sphere

Note: $|m| \leq l$

Spherical Harmonics Y_l^m

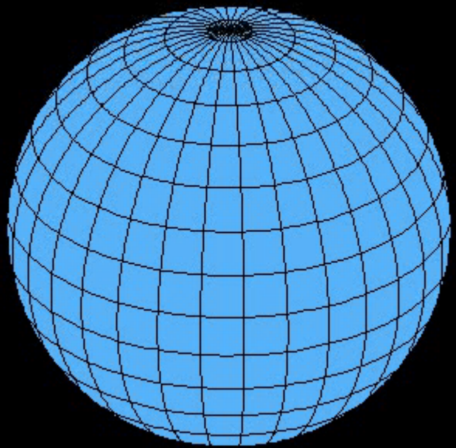
l – *angular degree: the number of nodes on the sphere*

$$k_h = \frac{\sqrt{l(l+1)}}{R}$$

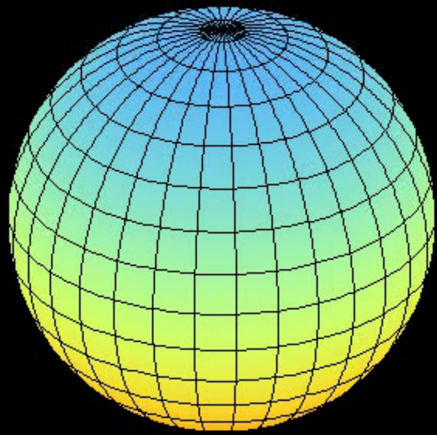
m - azimuthal order: $|m|$ =number of nodes along the equator

=> orientation on the sphere

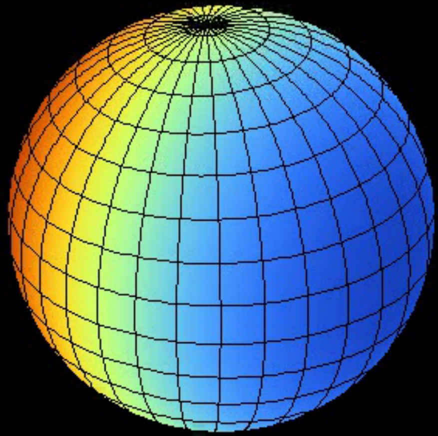
Note: $|m| \le l$



l=0



l=1
m=0



l=1
m=-1

Spherical Harmonics Y_l^m

l – *angular degree: the number of nodes on the sphere*

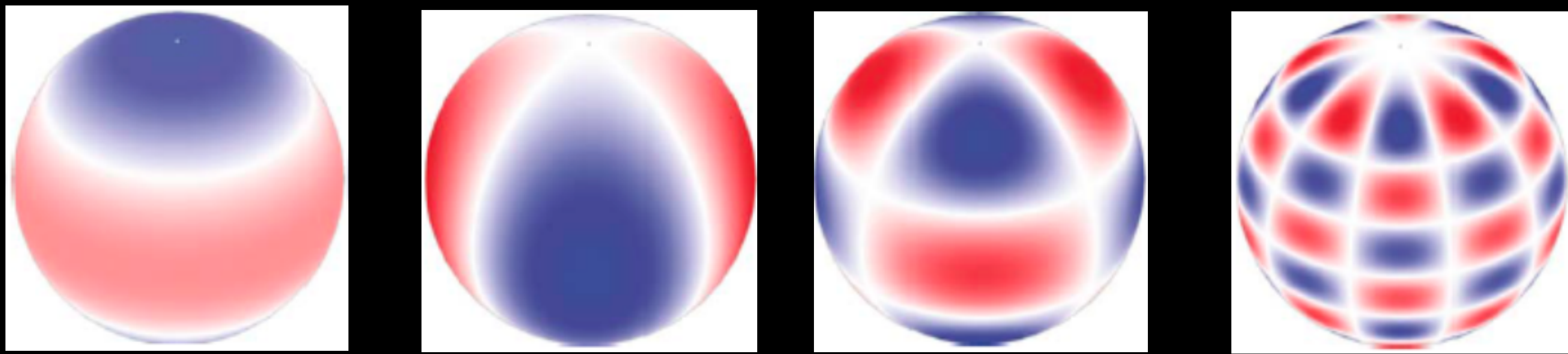
$$k_h = \frac{\sqrt{l(l+1)}}{R}$$

m - azimuthal order: $|m|$ =number of nodes along the equator

=> orientation on the sphere

Note: $|m| \le l$

adapted from Aerts et al. 2010



Spherical Harmonics Y_l^m

l – *angular degree: the number of nodes on the sphere*

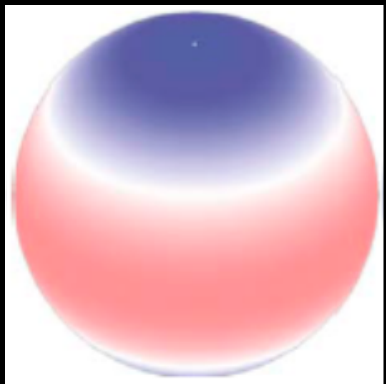
$$k_h = \frac{\sqrt{l(l+1)}}{R}$$

m - azimuthal order: $|m|$ =number of nodes along the equator

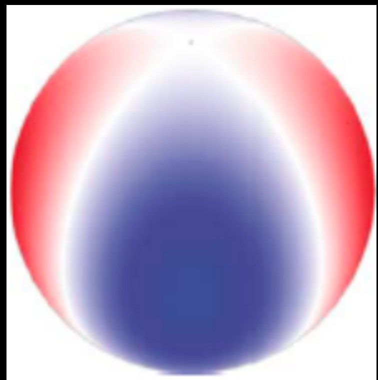
=> orientation on the sphere

Note: $|m| \leq l$

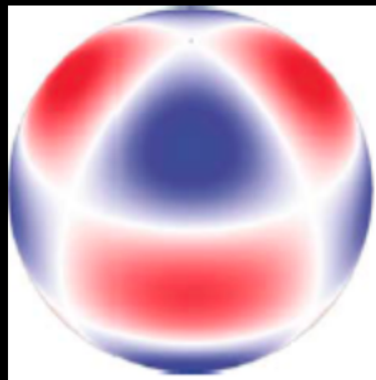
adapted from Aerts et al. 2010



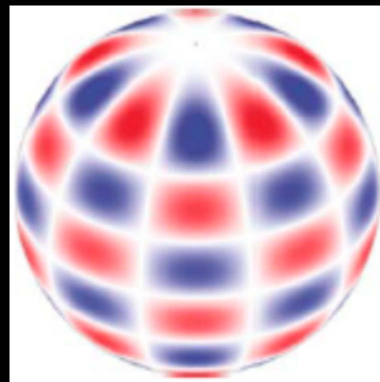
l=2
m=0



l=2
$|m|$=2



l=4
$|m|$=2



l=10
$|m|$=5

Solutions on a sphere

Consider the spherical coordinates (r,θ,φ)

Variables $(\varrho', p', \phi', \delta\vec{r})$ are function of: r, θ, φ, t

The equations admit solutions of the type:

$$p'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[p'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\rho'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[\rho'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\phi'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[\phi'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\delta\vec{\mathrm{r}}(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}\left\{\left[\xi_r(r)Y_l^m\hat{\mathrm{a}}_\mathrm{r} + \xi_h(r)\left(\frac{\partial Y_l^m}{\partial\theta}\hat{\mathrm{a}}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\hat{\mathrm{a}}_\phi\right)\right]\mathrm{e}^{-i\omega t}\right\}$$

Solutions on a sphere

Consider the spherical coordinates (r,θ,φ)

Variables $(\varrho', p', \phi', \delta\vec{r})$ are function of: r, θ, φ, t

The equations admit solutions of the type:

$$p'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[p'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\rho'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[\rho'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\phi'(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}[\phi'(r)Y_l^m(\theta,\varphi)\mathrm{e}^{-i\omega t}]$$

$$\delta\vec{\mathrm{r}}(\mathrm{r},\theta,\varphi,t) = \mathrm{Re}\left\{\left[\xi_r(r)Y_l^m\hat{\mathrm{a}}_\mathrm{r} + \xi_h(r)\left(\frac{\partial Y_l^m}{\partial\theta}\hat{\mathrm{a}}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\hat{\mathrm{a}}_\phi\right)\right]\mathrm{e}^{-i\omega t}\right\}$$

Equations for the depth dependent amplitudes

Substituting the solutions on the perturbed equations
… and after significant algebra

$$\frac{d\xi_r}{dr} = -\left(\frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr} + \frac{2}{r}\right)\xi_r + \left(\frac{S_l^{\,2}}{\omega^2} - 1\right)\frac{1}{c_0^{\,2}\rho_0}p' + \frac{l(l+1)}{r^2\omega^2}\phi'$$

$$\frac{dp'}{dr} = \rho_0(\omega^2 - N_0^2)\xi_r - \rho_0\frac{d\phi'}{dr} + \frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr}p'$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi'}{dr}\right) = 4\pi G\left(\frac{p'}{c_0^2} + \frac{\rho_0 N_0^2}{g_0}\xi_r\right) + \frac{l(l+1)}{r^2}\phi'$$

Equations for the depth dependent amplitudes

Substituting the solutions on the perturbed equations
… and after significant algebra

$$\frac{d\xi_r}{dr} = -\left(\frac{1}{\Gamma_{1,0} p_0}\frac{dp_0}{dr} + \frac{2}{r}\right)\xi_r + \left(\frac{S_l^{\,2}}{\omega^2} - 1\right)\frac{1}{c_0^{\,2}\rho_0}p' + \frac{l(l+1)}{r^2\omega^2}\phi'$$

$$\frac{dp'}{dr} = \rho_0(\omega^2 - N_0^2)\xi_r - \rho_0\frac{d\phi'}{dr} + \frac{1}{\Gamma_{1,0} p_0}\frac{dp_0}{dr}p'$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi'}{dr}\right) = 4\pi G\left(\frac{p'}{c_0^2} + \frac{\rho_0 N_0^2}{g_0}\xi_r\right) + \frac{l(l+1)}{r^2}\phi'$$

4 variables: ξ_r, p’, ϕ’, dϕ’/dr
4th order system

Note1: all derivatives are total derivatives because the functions depend on r only

Equations for the depth dependent amplitudes

Substituting the solutions on the perturbed equations
… and after significant algebra

$$\frac{d\xi_r}{dr} = -\left(\frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr} + \frac{2}{r}\right)\xi_r + \left(\frac{S_l^{\,2}}{\omega^2} - 1\right)\frac{1}{c_0^{\,2}\rho_0}p' + \frac{l(l+1)}{r^2\omega^2}\phi'$$

$$\frac{dp'}{dr} = \rho_0(\omega^2 - N_0^2)\xi_r - \rho_0\frac{d\phi'}{dr} + \frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr}p'$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi'}{dr}\right) = 4\pi G\left(\frac{p'}{c_0^2} + \frac{\rho_0 N_0^2}{g_0}\xi_r\right) + \frac{l(l+1)}{r^2}\phi'$$

4 variables: ξ_r, p’, ϕ’, dϕ’/dr
4th order system

Note2: equations depend on *l* but not on *m*, thus the eigenvalues *ω* cannot depend on *m*.

Equations for the depth dependent amplitudes

Substituting the solutions on the perturbed equations
… and after significant algebra

$$\frac{d\xi_r}{dr} = -\left(\frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr} + \frac{2}{r}\right)\xi_r + \left(\frac{S_l^{\,2}}{\omega^2} - 1\right)\frac{1}{c_0^{\,2}\rho_0}p' + \frac{l(l+1)}{r^2\omega^2}\phi'$$

$$\frac{dp'}{dr} = \rho_0(\omega^2 - N_0^2)\xi_r - \rho_0\frac{d\phi'}{dr} + \frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr}p'$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi'}{dr}\right) = 4\pi G\left(\frac{p'}{c_0^2} + \frac{\rho_0 N_0^2}{g_0}\xi_r\right) + \frac{l(l+1)}{r^2}\phi'$$

4 variables: ξ_r, p', ϕ', dϕ'/dr
4th order system

This system, together with the boundary conditions, forms an eigenvalue problem
=> Solving it provide the eigenvalues, ω, and eigenfunctions, ξ_r, p', ϕ', dϕ'/dr.

Equations for the depth dependent amplitudes

Substituting the solutions on the perturbed equations
… and after significant algebra

$$\frac{d\xi_r}{dr} = -\left(\frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr} + \frac{2}{r}\right)\xi_r + \left(\frac{S_l^2}{\omega^2} - 1\right)\frac{1}{c_0^{\ 2}\rho_0}p' + \frac{l(l+1)}{r^2\omega^2}\phi'$$

$$\frac{dp'}{dr} = \rho_0(\omega^2 - N_0^2)\xi_r - \rho_0\frac{d\phi'}{dr} + \frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr}p'$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi'}{dr}\right) = 4\pi G\left(\frac{p'}{c_0^2} + \frac{\rho_0 N_0^2}{g_0}\xi_r\right) + \frac{l(l+1)}{r^2}\phi'$$

S_l : Lamb frequency $\quad S_l^2 = \frac{l(l+1)}{r^2}c_0^2$

N_0 : Buoyancy frequency $\quad N_0^2 = g_0\left[\frac{1}{\Gamma_{1,0}}\frac{d\ln p_0}{dr} - \frac{d\ln \rho_0}{dr}\right]$

Equations for the depth dependent amplitudes

Substituting the solutions on the perturbed equations
… and after significant algebra

$$\frac{d\xi_r}{dr} = -\left(\frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr} + \frac{2}{r}\right)\xi_r + \left(\frac{S_l^2}{\omega^2} - 1\right)\frac{1}{c_0^{\,2}\rho_0}p' + \frac{l(l+1)}{r^2\omega^2}\phi'$$

$$\frac{dp'}{dr} = \rho_0(\omega^2 - N_0^2)\xi_r - \rho_0\frac{d\phi'}{dr} + \frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr}p'$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi'}{dr}\right) = 4\pi G\left(\frac{p'}{c_0^2} + \frac{\rho_0 N_0^2}{g_0}\xi_r\right) + \frac{l(l+1)}{r^2}\phi'$$

S_l : Lamb frequency $\quad S_l^2 = \frac{l(l+1)}{r^2}c_0^2$

N_0 : Buoyancy frequency $\quad N_0^2 = g_0\left[\frac{1}{\Gamma_{1,0}}\frac{d\ln p_0}{dr} - \frac{d\ln \rho_0}{dr}\right]$

$N_0^2 > 0 \Rightarrow \varrho_2^* > \varrho_2$

$N_0^2 < 0 \Rightarrow \varrho_2^* < \varrho_2$

ϱ_2^* $\quad \varrho_2$

ϱ_1^* $\quad \varrho_1$

Equations for the depth dependent amplitudes

Substituting the solutions on the perturbed equations
… and after significant algebra

$$\frac{d\xi_r}{dr} = -\left(\frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr}+\frac{2}{r}\right)\xi_r+\left(\frac{S_l^2}{\omega^2}-1\right)\frac{1}{c_0{}^2\rho_0}p'+\frac{l(l+1)}{r^2\omega^2}\phi'$$

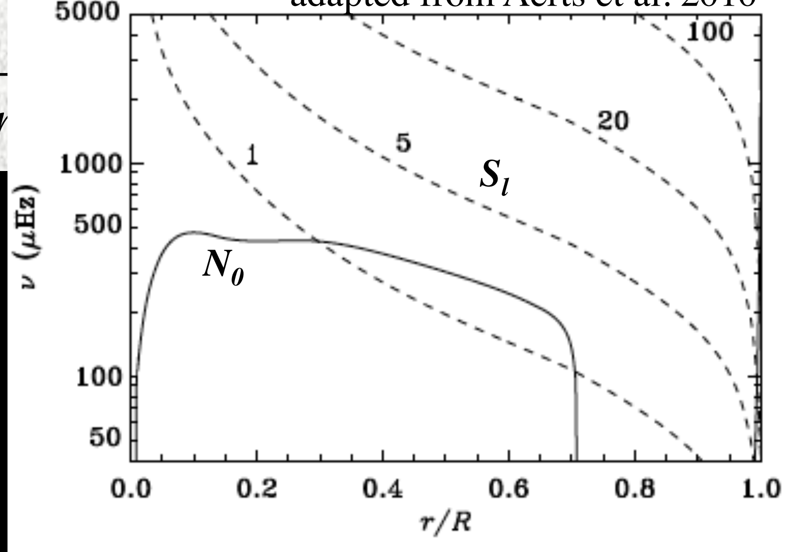
$$\frac{dp'}{dr} = \rho_0(\omega^2-N_0^2)\xi_r-\rho_0\frac{d\phi'}{dr}+\frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr}p'$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi'}{dr}\right) = 4\pi G\left(\frac{p'}{c_0^2}+\frac{\rho_0 N_0^2}{g_0}\xi_r\right)+\frac{l(l}{r}$$

S_l : Lamb frequency $\qquad S_l^2 = \frac{l(l+1)}{r^2}c_0^2$

N_0 : Buoyancy frequency $\qquad N_0^2 = g_0\left[\frac{1}{\Gamma_{1,0}}\frac{d\ln p_0}{dr}-\frac{d\ln\rho_0}{dr}\right]$

adapted from Aerts et al. 2010



Boundary conditions

Fourth order system => 4 boundary conditions required

- 2 conditions at r=0
- 2 condition at r=R

Boundary conditions

Fourth order system => 4 boundary conditions required

- 2 conditions at r=0
- 2 condition at r=R

Conditions at r=0

Obtained by imposing regularity of the solutions at the centre

Expand the equations near r=0 => find that

$p' \sim O(r^l)$; $\Phi' \sim O(r^l)$; $\xi_r \sim O(r^\alpha)$ with $\alpha=1$ for $l=0$

$\alpha=l-1$ for $l>0$

Boundary conditions

Fourth order system => 4 boundary conditions required

- 2 conditions at r=0
- 2 condition at r=R

Conditions at r=0

Obtained by imposing regularity of the solutions at the centre

Expand the equations near r=0 => find that

$p' \sim O(r^l)$; $\phi' \sim O(r^l)$; $\xi_r \sim O(r^\alpha)$ with α=1 for l=0

$\alpha = l-1$ for $l>0$

Consequently:

$$\frac{d\phi'}{dr} = \frac{l}{r}\phi' \quad ; \quad \frac{dp'}{dr} = \frac{l}{r}p' \quad ; \quad \frac{d\xi_r}{dr} = \frac{\alpha}{r}\xi_r$$

Boundary conditions

Fourth order system => 4 boundary conditions required

- 2 conditions at r=0
- 2 condition at r=R

Conditions at r=0

Obtained by imposing regularity of the solutions at the centre

Expand the equations near r=0 => find that

$p' \sim O(r^l)$; $\phi' \sim O(r^l)$; $\xi_r \sim O(r^\alpha)$ with α=1 for l=0

$\alpha = l-1$ for $l>0$

Consequently:

$$\boxed{\frac{d\phi'}{dr} = \frac{l}{r}\phi'} \; ; \; \boxed{\frac{dp'}{dr} = \frac{l}{r}p'} \; ; \; \frac{d\xi_r}{dr} = \frac{\alpha}{r}\xi_r$$

Boundary conditions

Fourth order system => 4 boundary conditions required

- 2 conditions at r=0
- 2 condition at r=R

Conditions at r=0

$$\frac{d\phi'}{dr} = \frac{l}{r}\phi' \quad ; \quad \frac{dp'}{dr} = \frac{l}{r}p'$$

Conditions at r=R

1^{st} condition: matching ϕ' and its derivative to solution for vacuum field

$\phi' \sim O\,(r^{-l-1})$

Boundary conditions

Fourth order system => 4 boundary conditions required

- 2 conditions at r=0
- 2 condition at r=R

Conditions at r=0

$$\frac{d\phi'}{dr} = \frac{l}{r}\phi' \quad ; \quad \frac{dp'}{dr} = \frac{l}{r}p'$$

Conditions at r=R

1st condition: matching ϕ' and its derivative to solution for vacuum field

$\phi' \sim O\,(r^{-l-1})$

$$\frac{d\phi'}{dr} = -\frac{(l+1)}{r}\phi'$$

Boundary conditions

Fourth order system => 4 boundary conditions required

- 2 conditions at r=0
- 2 condition at r=R

Conditions at r=0

$$\frac{d\phi'}{dr} = \frac{l}{r}\phi' \quad ; \quad \frac{dp'}{dr} = \frac{l}{r}p'$$

Conditions at r=R

1st condition: matching ϕ' and its derivative to solution for vacuum field

$\phi' \sim O\,(r^{-l-1})$

$$\frac{d\phi'}{dr} = -\frac{(l+1)}{r}\phi'$$

2nd condition: depends on how the atmosphere is treated

e.g. assuming free surface => $\delta p'$=0

$$p' + \xi_r \frac{dp_0}{dr} = 0$$

(But this is not adequate for a real star!)

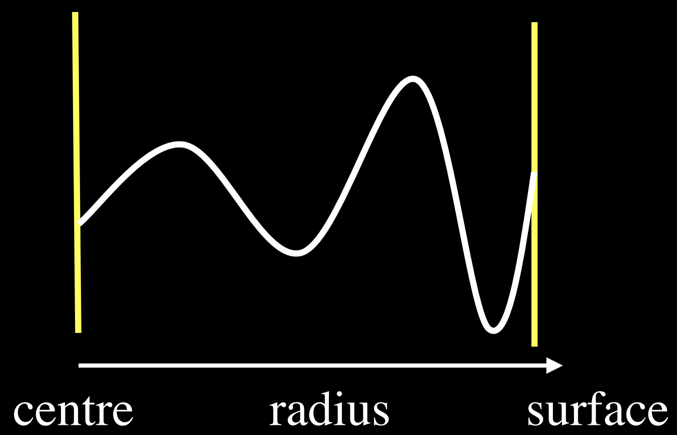
A better option is to make the numerical solutions match onto the analytical solutions for an isothermal atmosphere.

Eigenvalue problem

We reduced the problem to 1D

Equations + boundary conditions
=> admit non-trivial solutions only for a discrete values of frequencies

This set of frequencies is numbered by an integer *n, the radial order*



Eigenvalue problem

In summary: eigenfrequencies are discrete and characterized by three quantum numbers:

$$\boxed{\omega=\omega(n,l,m)}$$

n –radial order: |n| related to the number of nodes along the radial direction

l – angular degree: the number of nodes on the sphere

m - azimuthal order: $|m|$ =number of nodes along the equator

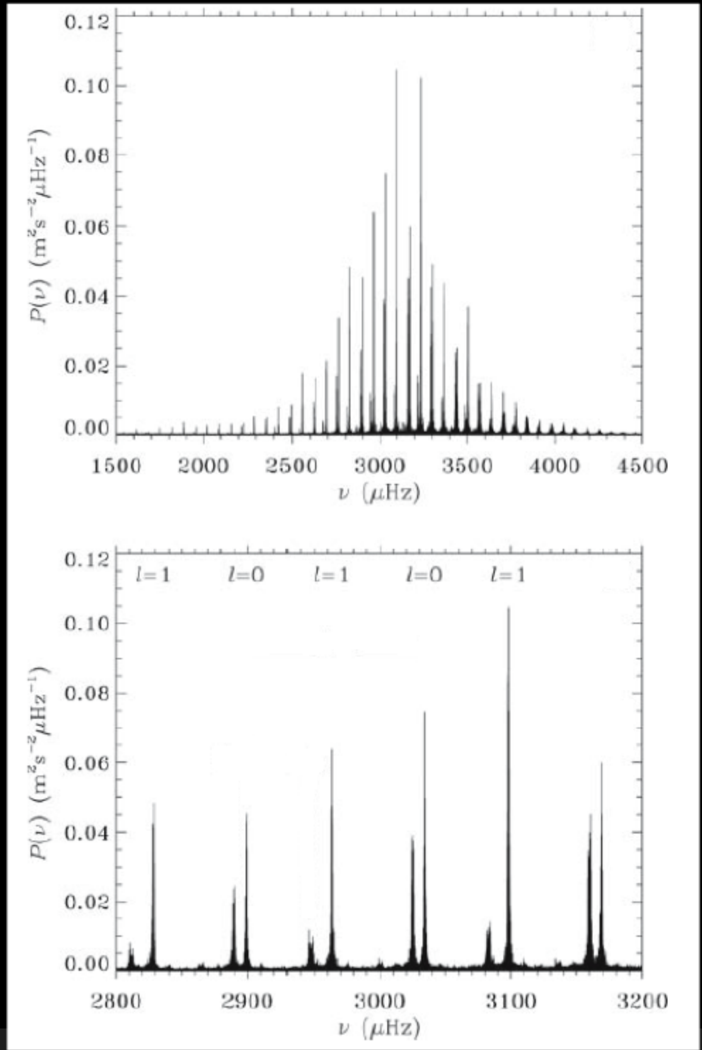
=> orientation on the sphere

Eigenvalue problem

In summary: eigenfrequencies are discrete and characterized by three quantum numbers:

$$\boxed{\omega=\omega(n,l,m)}$$

Adapted from Cunha et al 2007 (Bison data)



Equations for the depth dependent amplitudes

$$\frac{d\xi_r}{dr} = -\left(\frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr} + \frac{2}{r}\right)\xi_r + \left(\frac{S_l^{\,2}}{\omega^2} - 1\right)\frac{1}{c_0^{\,2}\rho_0}p' + \frac{l(l+1)}{r^2\omega^2}\phi'$$

$$\frac{dp'}{dr} = \rho_0(\omega^2 - N_0^2)\xi_r - \rho_0\frac{d\phi'}{dr} + \frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr}p'$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi'}{dr}\right) = 4\pi G\left(\frac{p'}{c_0^2} + \frac{\rho_0 N_0^2}{g_0}\xi_r\right) + \frac{l(l+1)}{r^2}\phi'$$

Equations depend on l, but not on m

=> In a spherically symmetric star, the eigenvalues are independent of m

Equations for the depth dependent amplitudes

$$\frac{d\xi_r}{dr} = -\left(\frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr} + \frac{2}{r}\right)\xi_r + \left(\frac{S_l^{\,2}}{\omega^2} - 1\right)\frac{1}{c_0^{\,2}\rho_0}p' + \frac{l(l+1)}{r^2\omega^2}\phi'$$

$$\frac{dp'}{dr} = \rho_0(\omega^2 - N_0^2)\xi_r - \rho_0\frac{d\phi'}{dr} + \frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr}p'$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi'}{dr}\right) = 4\pi G\left(\frac{p'}{c_0^2} + \frac{\rho_0 N_0^2}{g_0}\xi_r\right) + \frac{l(l+1)}{r^2}\phi'$$

Equations depend on l, but not on m

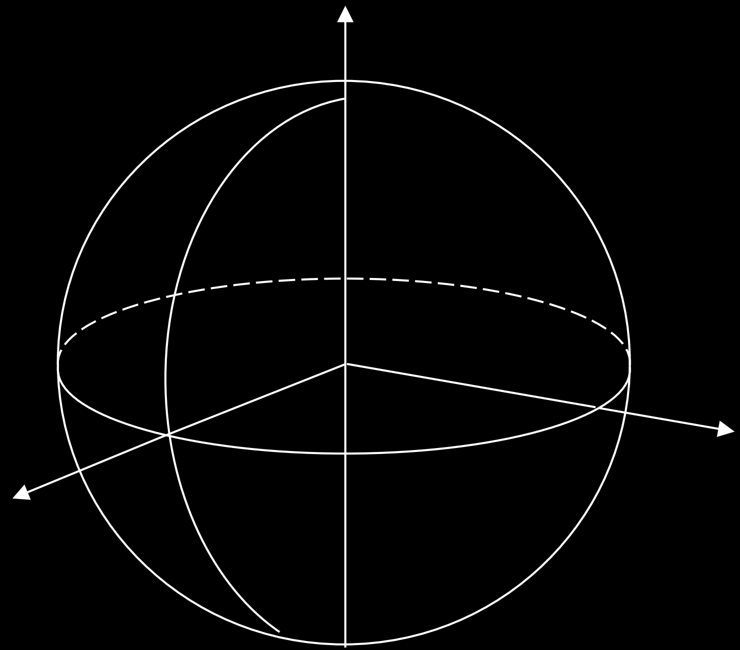
=> In a spherically symmetric star, the eigenvalues are independent of m

$$\boxed{\omega = \omega(n, l, \cancel{m})}$$

Note: That is not the case if the star rotates or has a magnetic field, braking the symmetry.

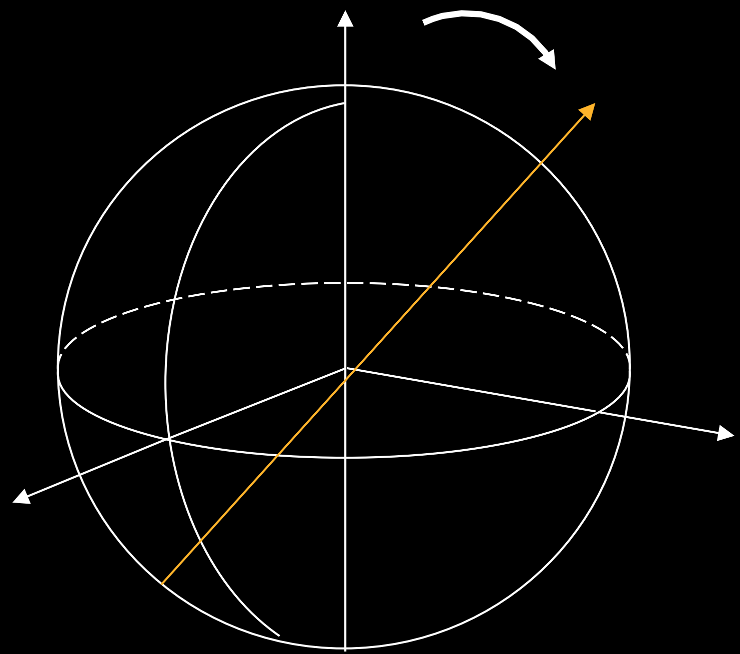
Waves in a spherically symmetric star

$$\omega^2 \widetilde{\delta f} = \mathcal{L}(\widetilde{\delta f})$$



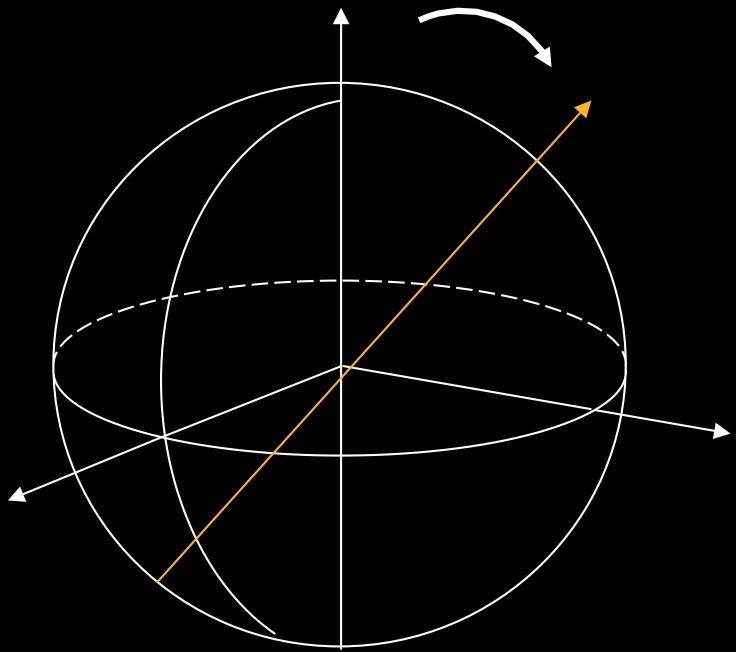
Waves in a spherically symmetric star

$$\omega^2 \widetilde{\delta f} = \mathcal{L}(\widetilde{\delta f})$$



Waves in a spherically symmetric star

$$\omega^2 \widetilde{\delta f} = \mathcal{L}(\widetilde{\delta f})$$



$$\boxed{\omega = \omega(n, l, \cancel{m})}$$

One eigenmode $\leftrightarrow$ any combination

$$\widetilde{\delta f} = \sum_m a_m Y_l^m$$

Trapping of the oscillations

Trapping of oscillations

The full solutions must be obtained numerically. However, under particular approximations, approximate analytical solutions can be derived.

Trapping of oscillations

The full solutions must be obtained numerically. However, under particular approximations, approximate analytical solutions can be derived.

The Cowling approximation

Neglect the perturbation to the gravitational potential, ϕ'

=> reduces the system to 2nd order

Valid when l is large or $|n|$ is large

$$\vec{g}' = \nabla\Phi'$$

$$\Phi' = G\int_V \frac{\rho'(\vec{r}',t)}{|\vec{r}-\vec{r}'|}dV$$

Trapping of oscillations

The full solutions must be obtained numerically. However, under particular approximations, approximate analytical solutions can be derived.

The Cowling approximation

Neglect the perturbation to the gravitational potential, ϕ'
=> reduces the system to 2nd order

Valid when l is large or $|n|$ is large

$$\frac{d\xi_r}{dr} = -\left(\frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr} + \frac{2}{r}\right)\xi_r + \left(\frac{S_l^{\,2}}{\omega^2} - 1\right)\frac{1}{c_0^{\,2}\rho_0}p' + \cancel{\frac{l(l+1)}{r^2\omega^2}}\phi'$$

$$\frac{dp'}{dr} = \rho_0(\omega^2 - N_0^2)\xi_r - \cancel{\rho_0\frac{d\phi'}{dr}} + \frac{1}{\Gamma_{1,0}p_0}\frac{dp_0}{dr}p'$$

$$\frac{1}{r^2}\frac{d}{dr}\cancel{\left(r^2\frac{d\phi'}{dr}\right) = 4\pi G\left(\frac{p'}{c_0^2} + \frac{\rho_0 N_0^2}{g_0}\xi_r\right) +} \frac{l(l+1)}{r^2}\phi'$$

Trapping of oscillations

The full solutions must be obtained numerically. However, under particular approximations, approximate analytical solutions can be derived.

The Cowling approximation

Neglect the perturbation to the gravitational potential, ϕ'

=> reduces the system to 2nd order

Valid when l is large or $|n|$ is large

$$\frac{d\xi_r}{dr} = -\left(\frac{1}{\Gamma_{1,0} p_0}\frac{dp_0}{dr} + \frac{2}{r}\right)\xi_r + \left(\frac{S_l^{\,2}}{\omega^2} - 1\right)\frac{1}{c_0^{\,2}\rho_0} p'$$

$$\frac{dp'}{dr} = \rho_0(\omega^2 - N_0^2)\xi_r + \frac{1}{\Gamma_{1,0} p_0}\frac{dp_0}{dr} p'$$

2 variables: ξ_r, p'

2nd order system

Trapping of oscillations

Following Deubner and Gough 1984

- Work under Cowling approximation
- Assume that locally oscillations can be treated as in a plane-parallel layer under constant gravity (i.e., neglect derivatives of g and r)

(See also, Gough 93)

Trapping of oscillations

Following Deubner and Gough 1984

- Work under Cowling approximation
- Assume that locally oscillations can be treated as in a plane-parallel layer under constant gravity (i.e., neglect derivatives of g and r)

- Define the new variable:

$$X = c_0^{\ 2} \rho_0^{\ 1/2} \nabla \cdot \delta\vec{r}$$

Trapping of oscillations

Following Deubner and Gough 1984

- Work under Cowling approximation
- Assume that locally oscillations can be treated as in a plane-parallel layer under constant gravity (i.e., neglect derivatives of g and r)

- Define the new variable:

$$X = c_0^2 \rho_0^{1/2} \nabla \cdot \delta\vec{r}$$

In terms of the new variable the 2nd order system of equations can be reduced to a single 2nd order wave equation:

$$\frac{d^2 X}{dr^2} + k_r^2 X = 0$$

Where k_r is the local radial wavenember

Trapping of oscillations

Recall the solutions of the wave equation with constant k

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

Trapping of oscillations

Recall the solutions of the wave equation with constant k

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

General solution is: $$y = Ae^{ikx} + Be^{-ikx}$$

where A and B are complex constants

Trapping of oscillations

Recall the solutions of the wave equation with constant k

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

General solution is: $$y = Ae^{ikx} + Be^{-ikx}$$

where A and B are complex constants

- $k^2 > 0$ => k is real ; $\mathrm{Re}\{y\} = a\cos kx + b\sin kx$

  => *oscillatory behaviour*

- $k^2 < 0$ => $k = i\,|k|$; $\mathrm{Re}\{y\} = ae^{-|k|x} + be^{|k|x}$

  => *exponential grow or decay*

Trapping of oscillations

In the star k_r is not constant!

$$\frac{d^2 X}{dr^2} + k_r^2 X = 0$$

Trapping of oscillations

In the star k_r is not constant!

$$\frac{d^2X}{dr^2} + k_r^2 X = 0$$

$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2} - 1\right) + \omega^2 - \omega_c^2\right]$$

$$\omega_c^2 = \frac{c_0^{\,2}}{4H^2}\left(1 - 2\frac{dH}{dr}\right)$$

$$H^{-1} = -\frac{d\ln\rho}{dr}$$

Trapping of oscillations

In the star k_r is not constant!

$$\frac{d^2X}{dr^2} + k_r^2 X = 0$$

$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2} - 1\right) + \omega^2 - \omega_c^2\right]$$

$$\omega_c^2 = \frac{c_0^{\,2}}{4H^2}\left(1 - 2\frac{dH}{dr}\right)$$

$$H^{-1} = -\frac{d\ln\rho}{dr}$$

Trapping of oscillations

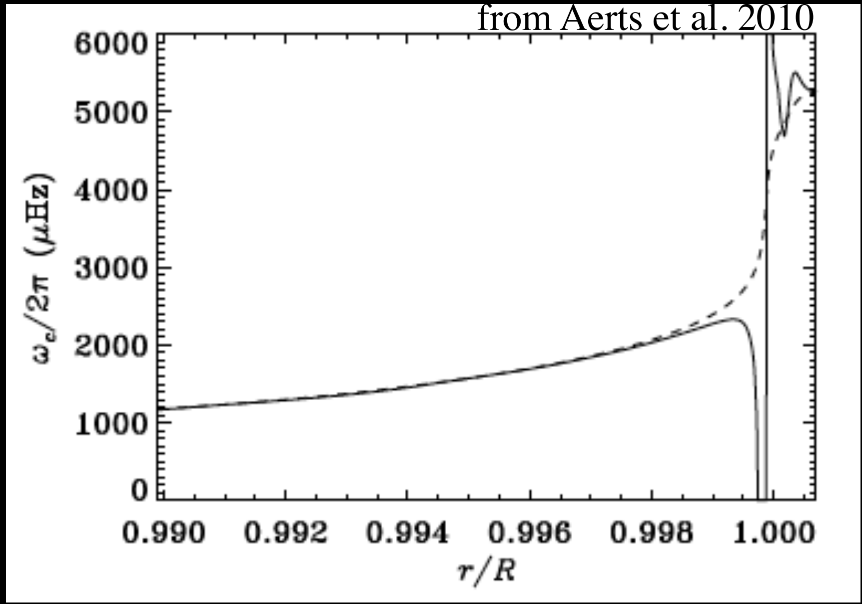
In the star k_r is not constant!

$$\frac{d^2X}{dr^2} + k_r^2 X = 0$$

$$k_r^2 = \frac{1}{c_0^2}\left[S_l^2\left(\frac{N_0^2}{\omega^2} - 1\right) + \omega^2 - \omega_c^2\right]$$

$$\omega_c^2 = \frac{c_0^2}{4H^2}\left(1 - 2\frac{dH}{dr}\right)$$

$$H^{-1} = -\frac{d\ln\rho}{dr}$$



Trapping of oscillations

In the star k_r is not constant!

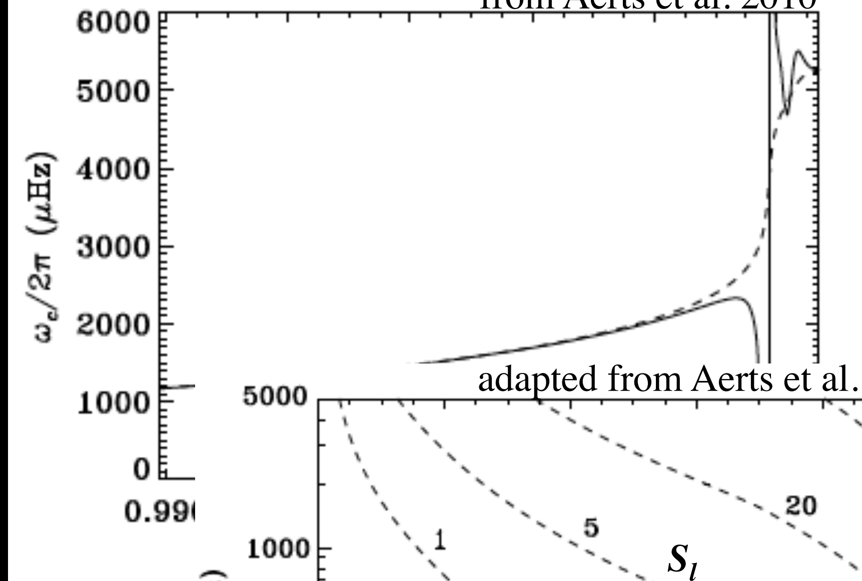
$$\frac{d^2X}{dr^2} + k_r^2 X = 0$$

$$k_r^2 = \frac{1}{c_0^2}\left[S_l^2\left(\frac{N_0^2}{\omega^2} - 1\right) + \omega^2 - \omega_c^2\right]$$

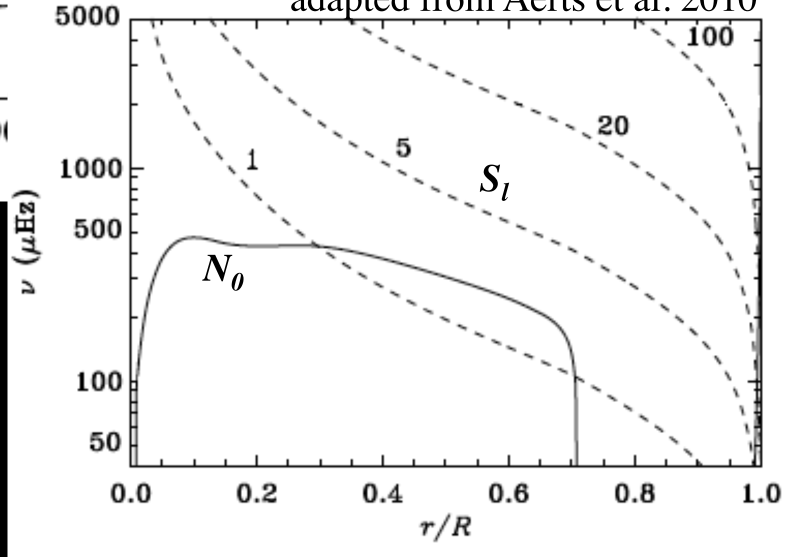
$$\omega_c^2 = \frac{c_0^2}{4H^2}\left(1 - 2\frac{dH}{dr}\right)$$

$$H^{-1} = -\frac{d\ln\rho}{dr}$$

from Aerts et al. 2010



adapted from Aerts et al. 2010



These 3 characteristic frequencies will play a fundamental role in deciding where modes propagate and where they are evanescent.

Trapping of oscillations

$$\frac{d^2X}{dr^2} + k_r^2 X = 0$$

$$k_r^2 = \frac{1}{c_0^2}\left[S_l^2\left(\frac{N_0^2}{\omega^2} - 1\right) + \omega^2 - \omega_c^2\right]$$

What are the regions where: $k_r^2 > 0$ (oscillatory behaviour) ?

$k_r^2 < 0$ (exponentially decaying) ?

Trapping of oscillations

$$\frac{d^2X}{dr^2} + k_r^2 X = 0$$

$$k_r^2 = \frac{1}{c_0{}^2}\left[S_l^2\left(\frac{N_0^2}{\omega^2} - 1\right) + \omega^2 - \omega_c^2\right]$$

What are the regions where: $k_r{}^2 > 0$ (oscillatory behaviour) ?

$k_r{}^2 < 0$ (exponentially decaying) ?

Find the turning points of the equation, where $k_r{}^2 = 0$

$$\omega_{l\pm}^2 = \frac{1}{2}\left(S_l^2 + \omega_c^2\right) \pm \frac{1}{2}\sqrt{\left(S_l^2 + \omega_c^2\right)^2 - 4S_l^2 N_0^2}$$

Trapping of oscillations

$$\frac{d^2X}{dr^2} + k_r^2 X = 0$$

$$k_r^2 = \frac{1}{c_0^{\ 2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2} - 1\right) + \omega^2 - \omega_c^2\right]$$

What are the regions where: $k_r^2 > 0$ (oscillatory behaviour) ?
$k_r^2 < 0$ (exponentially decaying) ?

Find the turning points of the equation, where $k_r^2 = 0$

$$\omega_{l\pm}^2 = \frac{1}{2}\left(S_l^2 + \omega_c^2\right) \pm \frac{1}{2}\sqrt{\left(S_l^2 + \omega_c^2\right)^2 - 4S_l^2 N_0^2}$$

Thus, we can rewrite: $$k_r^2 = \frac{1}{c_0^{\ 2}}\left[\omega^2 - \omega_{l+}^2\right]\left[\omega^2 - \omega_{l-}^2\right]$$

Trapping of oscillations

$$\frac{d^2X}{dr^2} + k_r^2 X = 0$$

$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2} - 1\right) + \omega^2 - \omega_c^2\right]$$

What are the regions where: $k_r^2 > 0$ (oscillatory behaviour) ?

$k_r^2 < 0$ (exponentially decaying) ?

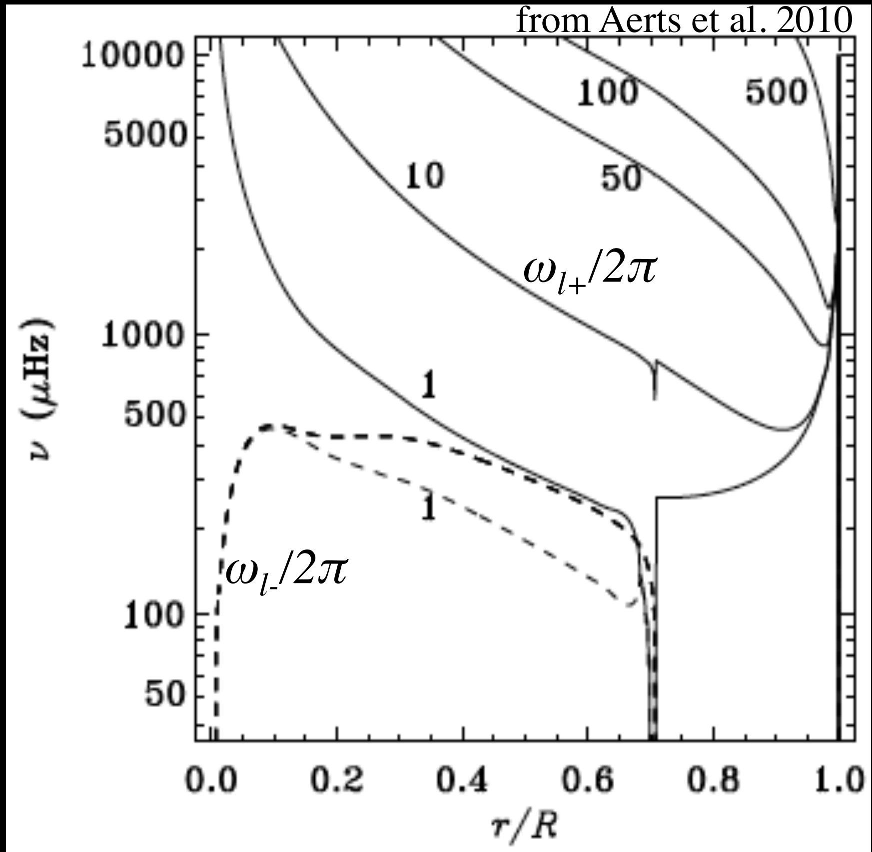
Find the turning points of the equation, where $k_r^2 = 0$

$$\omega_{l\pm}^2 = \frac{1}{2}\left(S_l^2 + \omega_c^2\right) \pm \frac{1}{2}\sqrt{\left(S_l^2 + \omega_c^2\right)^2 - 4S_l^2 N_0^2}$$

Thus, we can rewrite: $k_r^2 = \frac{1}{c_0^{\,2}}\left[\omega^2 - \omega_{l+}^2\right]\left[\omega^2 - \omega_{l-}^2\right]$

- Modes propagate where $k_r^2 > 0$ => $\omega > \omega_{l+}$ or $\omega < \omega_{l-}$
- Modes are evanescent where $k_r^2 < 0$ => $\omega_{l-} < \omega < \omega_{l+}$

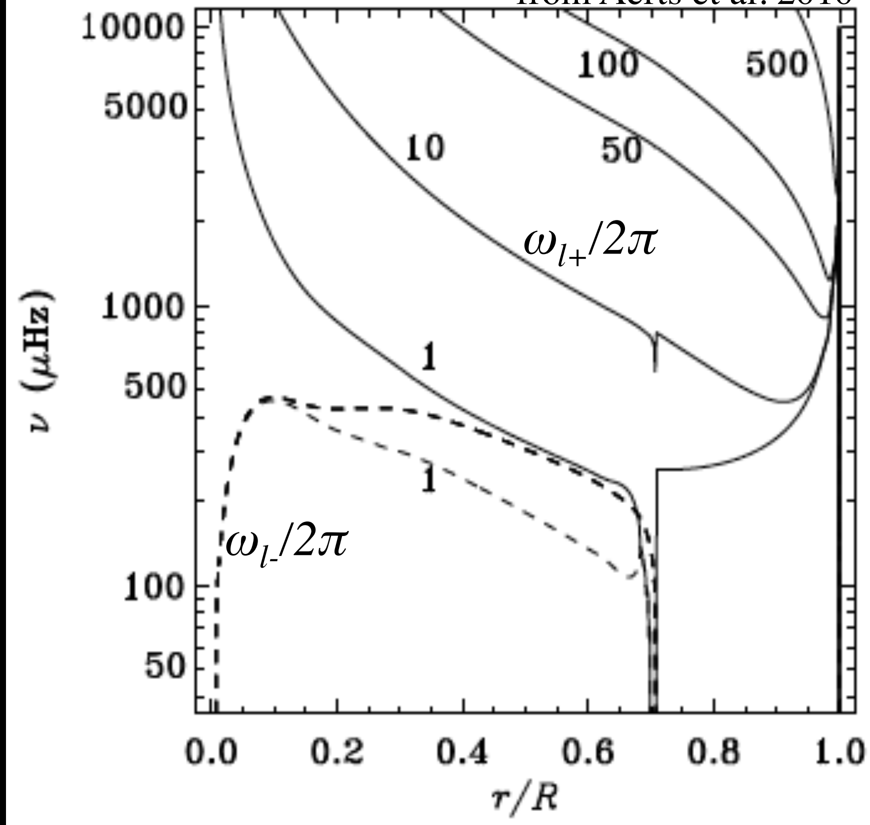
Trapping of oscillations

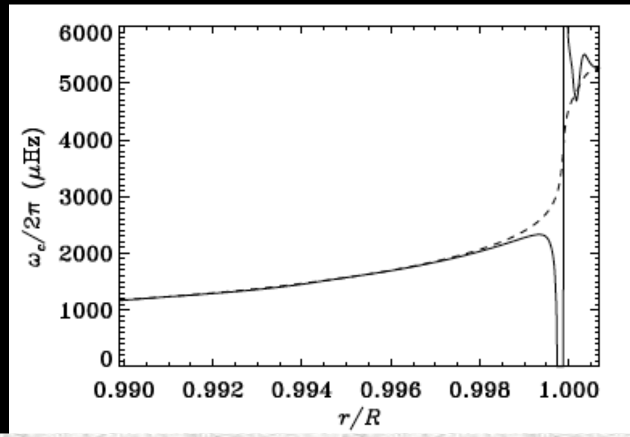


- Modes propagate where $k_r^2 > 0$ => $\omega > \omega_{l+}$ or $\omega < \omega_{l-}$
- Modes are evanescent where $k_r^2 < 0$ => $\omega_{l-} < \omega < \omega_{l+}$

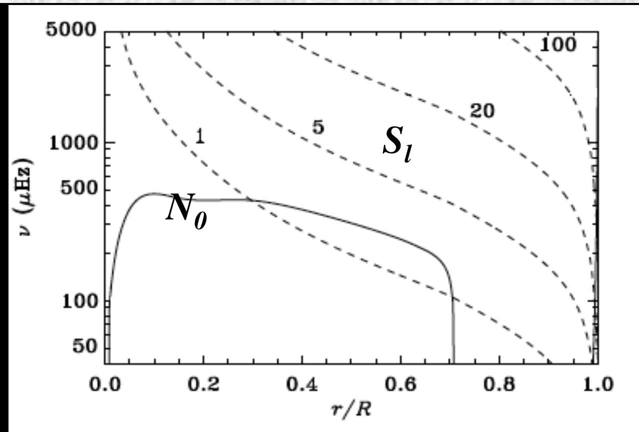
Trapping of oscillations

from Aerts et al. 2010



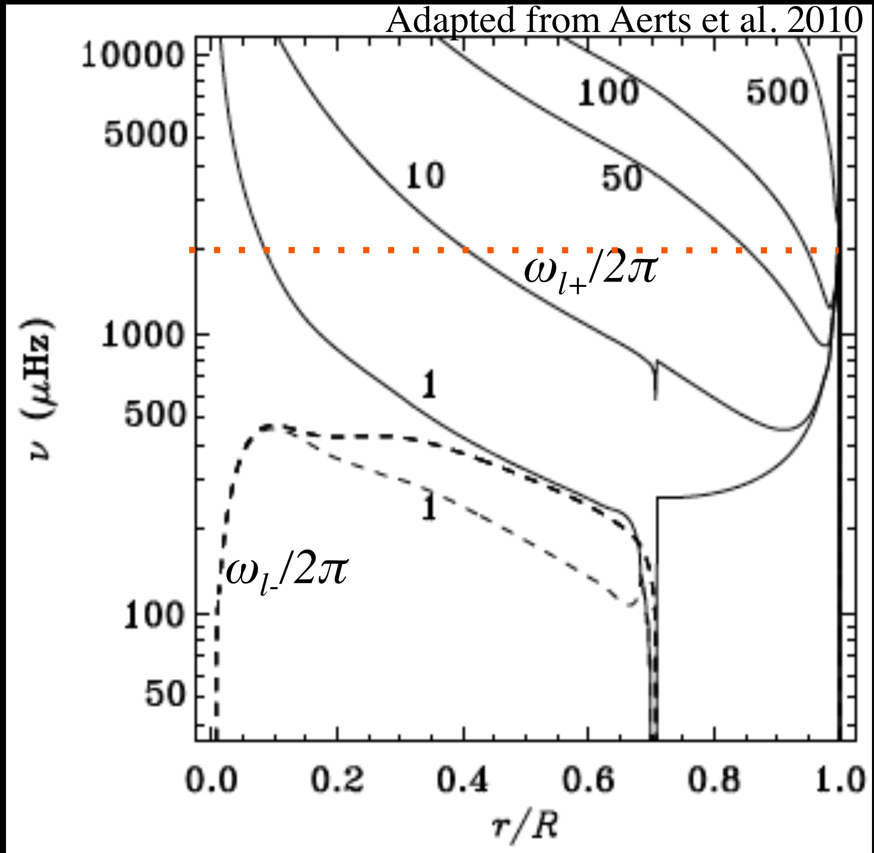


$$\omega_{l\pm}^2 = \frac{1}{2}\left(S_l^2 + \omega_c^2\right) \pm \frac{1}{2}\sqrt{\left(S_l^2 + \omega_c^2\right)^2 - 4S_l^2 N_0^2}$$



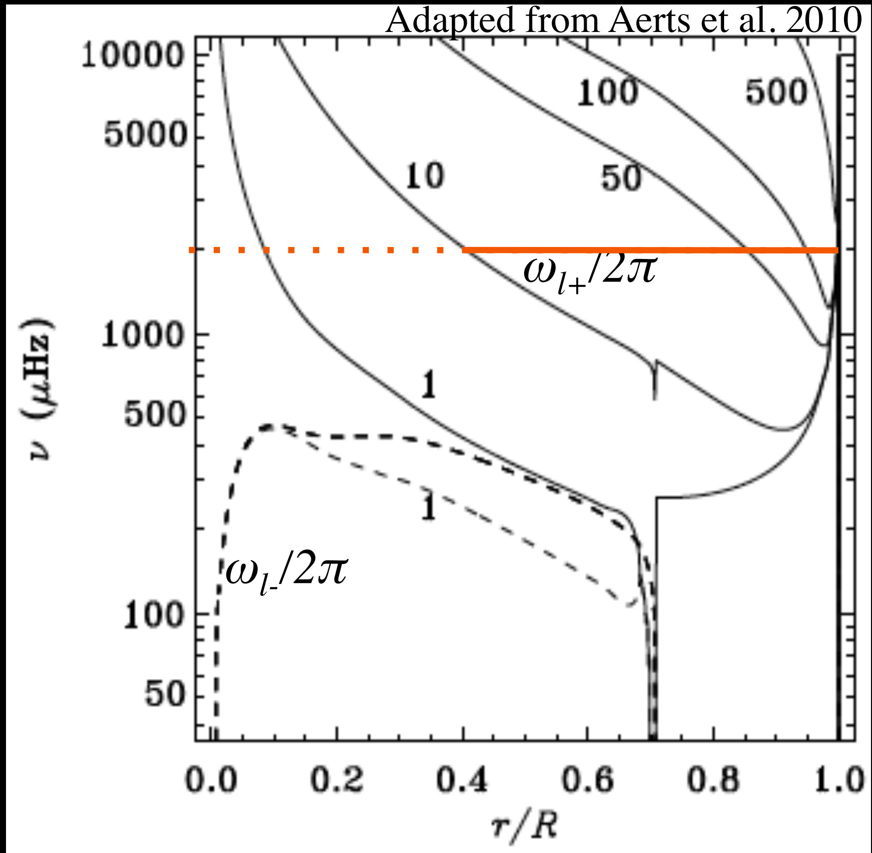
- Modes propagate where $k_r^2 > 0$ => $\omega > \omega_{l+}$ or $\omega < \omega_{l-}$
- Modes are evanescent where $k_r^2 < 0$ => $\omega_{l-} < \omega < \omega_{l+}$

Trapping of oscillations



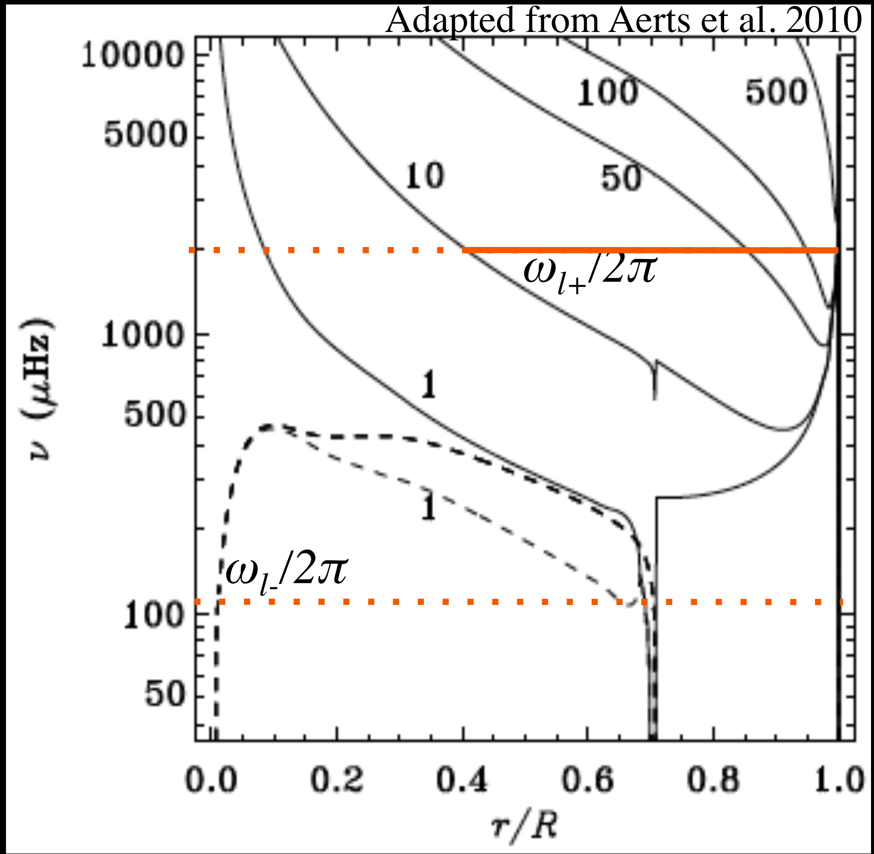
- Modes propagate where $k_r^2 > 0$ => $\omega > \omega_{l+}$ or $\omega < \omega_{l-}$
- Modes are evanescent where $k_r^2 < 0$ => $\omega_{l-} < \omega < \omega_{l+}$

Trapping of oscillations



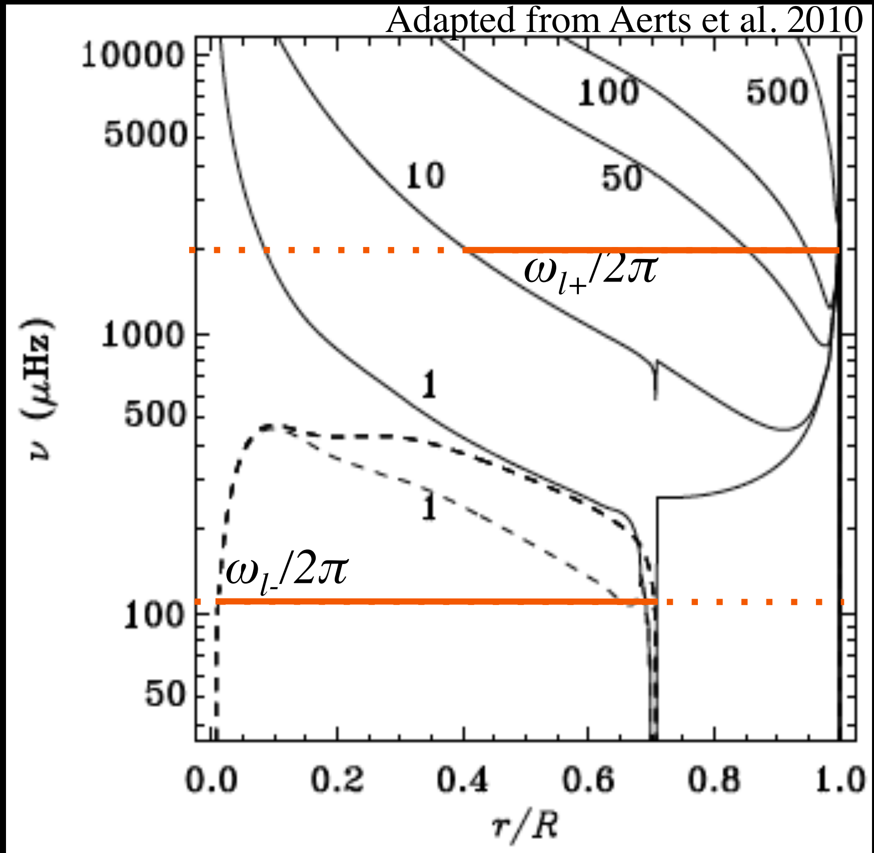
- Modes propagate where $k_r^2 > 0$ => $\omega > \omega_{l+}$ or $\omega < \omega_{l-}$
- Modes are evanescent where $k_r^2 < 0$ => $\omega_{l-} < \omega < \omega_{l+}$

Trapping of oscillations



- Modes propagate where $k_r^2 > 0$ => $\omega > \omega_{l+}$ or $\omega < \omega_{l-}$
- Modes are evanescent where $k_r^2 < 0$ => $\omega_{l-} < \omega < \omega_{l+}$

Trapping of oscillations



- Modes propagate where $k_r^2 > 0$ => $\omega > \omega_{l+}$ or $\omega < \omega_{l-}$
- Modes are evanescent where $k_r^2 < 0$ => $\omega_{l-} < \omega < \omega_{l+}$

A closer look at the solutions

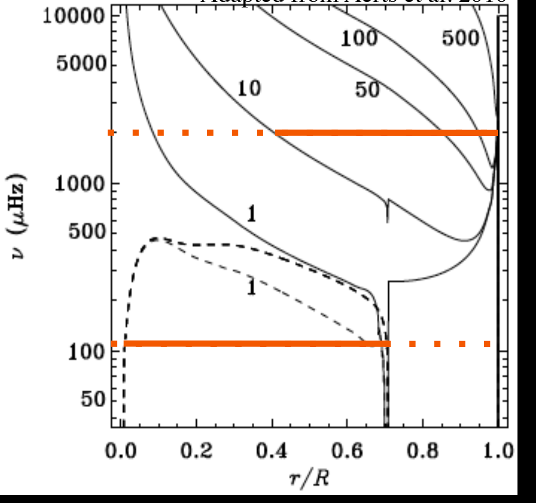
Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- High frequency modes $\omega^2 >> N_0^2$

Adapted from Aerts et al. 2010



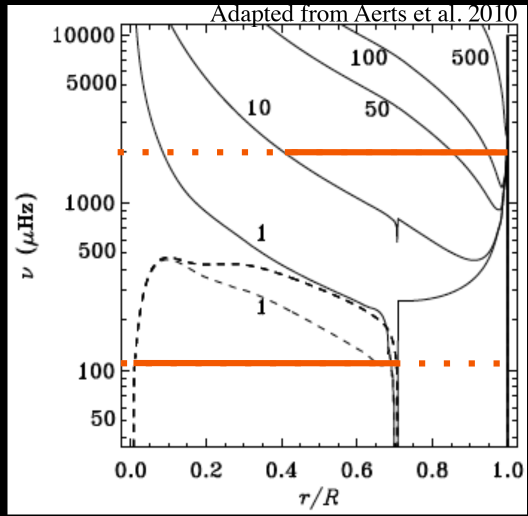
Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- High frequency modes $\omega^2 >> N_0^2$

Except near the surface $k_r^2 \approx \frac{\omega^2 - S_l^2}{c_0^{\,2}} = \frac{\omega^2}{c_0^{\,2}} - \frac{l(l+1)}{r^2}$



Trapping of oscillations

A closer look at the two families of solutions

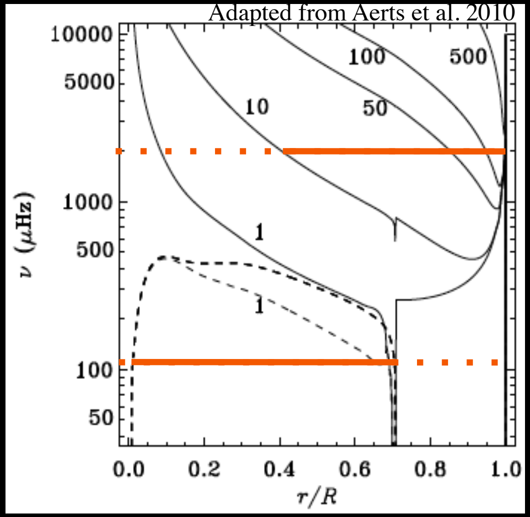
$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- High frequency modes $\omega^2 >> N_0^2$

Except near the surface

$$k_r^2 \approx \frac{\omega^2 - S_l^2}{c_0^{\,2}} = \frac{\omega^2}{c_0^{\,2}} - \frac{l(l+1)}{r^2}$$

$k_h^{\,2}$



Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

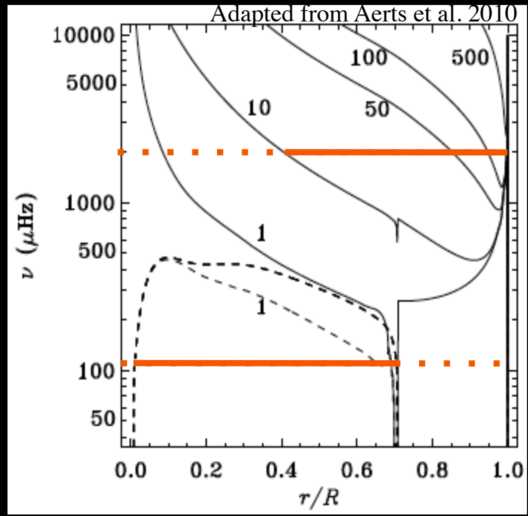
- High frequency modes $\omega^2 >> N_0^2$

Except near the surface

$$k_r^2 \approx \frac{\omega^2 - S_l^2}{c_0^{\,2}} = \frac{\omega^2}{c_0^{\,2}} - \frac{l(l+1)}{r^2}$$

k_h^2

$$k^2 \equiv k_r^2 + k_h^2 \approx \frac{\omega^2}{c_0^{\,2}}$$



Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- High frequency modes $\omega^2 >> N_0^2$

Except near the surface

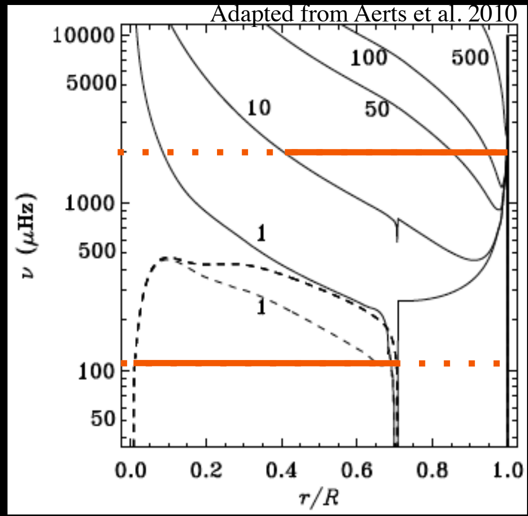
$$k_r^2 \approx \frac{\omega^2 - S_l^2}{c_0^{\,2}} = \frac{\omega^2}{c_0^{\,2}} - \frac{l(l+1)}{r^2}$$

$k_h^{\,2}$

$$k^2 \equiv k_r^2 + k_h^2 \approx \frac{\omega^2}{c_0^{\,2}}$$

$$\omega \approx c_0 k$$

Dispersion relation for acoustic wave!



Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^2}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- High frequency modes $\omega^2 >> N_0^2$

Except near the surface

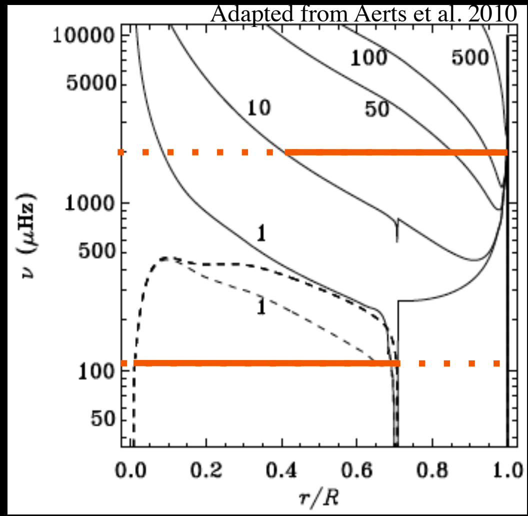
$$k_r^2 \approx \frac{\omega^2 - S_l^2}{c_0^2} = \frac{\omega^2}{c_0^2} - \frac{l(l+1)}{r^2}$$

k_h^2

$$k^2 \equiv k_r^2 + k_h^2 \approx \frac{\omega^2}{c_0^2}$$

$$\omega \approx c_0 k$$

Dispersion relation for acoustic wave!



ω increases as k increases

=> the radial order n increases with the frequency

Trapping of oscillations

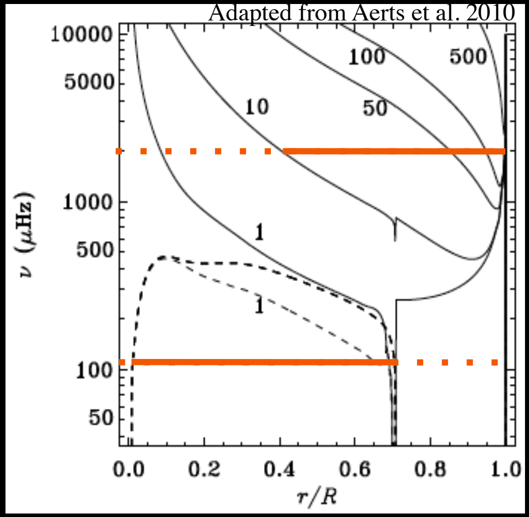
A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^{\ 2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- High frequency modes $\omega^2 >> N_0^2$

Except near the surface $k_r^2 \approx \frac{\omega^2 - S_l^2}{c_0^{\ 2}} = \frac{\omega^2}{c_0^{\ 2}} - \frac{l(l+1)}{r^2}$

Lower turning point



Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^2}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

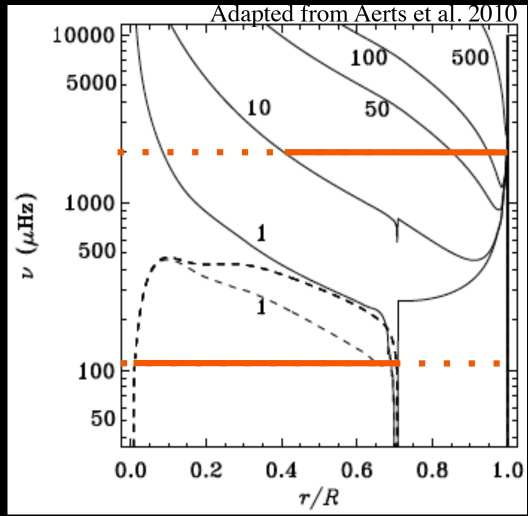
- High frequency modes $\omega^2 >> N_0^2$

Except near the surface

$$k_r^2 \approx \frac{\omega^2 - S_l^2}{c_0^2} = \frac{\omega^2}{c_0^2} - \frac{l(l+1)}{r^2}$$

Lower turning point $\omega^2 = S_l^2$

$$r_{1,l} = \frac{\sqrt{l(l+1)}c_0}{\omega}$$



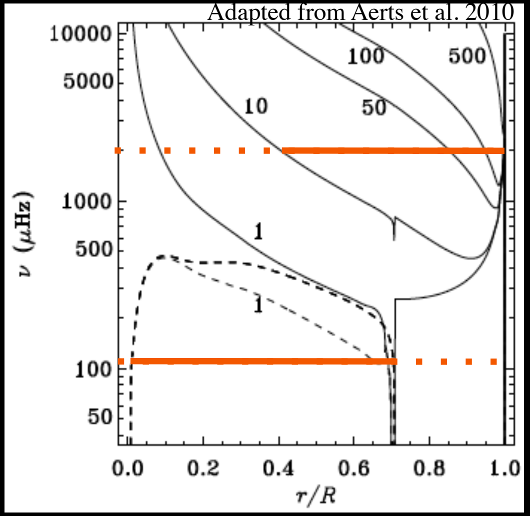
Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^2}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- High frequency modes $\omega^2 >> N_0^2$

Except near the surface $k_r^2 \approx \frac{\omega^2 - S_l^2}{c_0^2} = \frac{\omega^2}{c_0^2} - \frac{l(l+1)}{r^2}$



Lower turning point $\omega^2 = S_l^2$

$$r_{1,l} = \frac{\sqrt{l(l+1)}c_0}{\omega}$$

$r_{1,l}$ increases as l increases
=> larger degree modes have shallower cavities

For fixed l: $r_{1,l}$ increases as ω increases
=> higher frequency modes propagate deeper, for fixed degree

Trapping of oscillations

A closer look at the two families of solutions

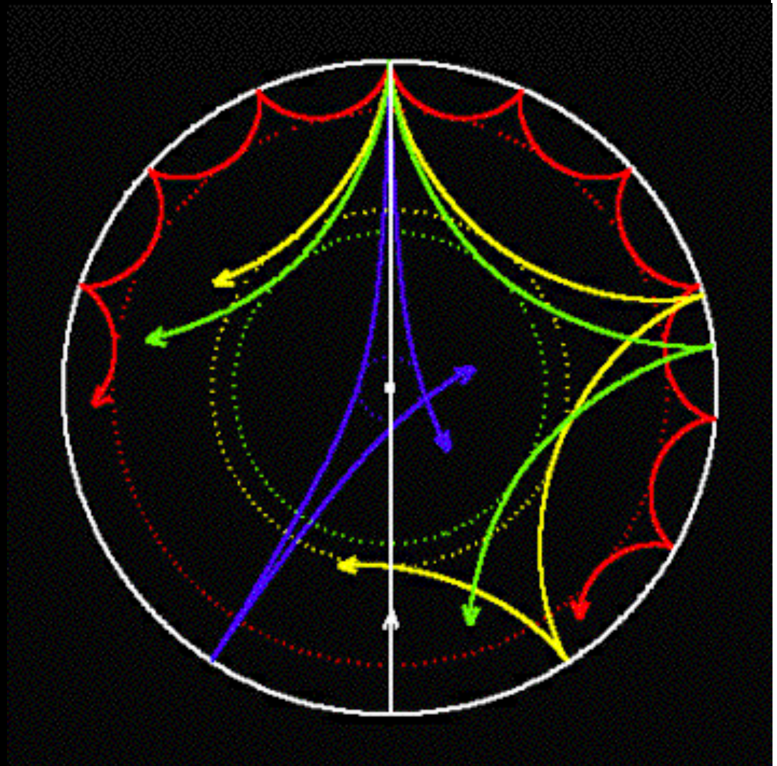
➢ High frequency modes $\omega^2 >> N_0^2$

Except near the surface

$$k_r^2 \approx \frac{\omega^2 - S_l^2}{c_0^2} = \frac{\omega^2}{c_0^2} - \frac{l(l+1)}{r^2}$$

Lower turning point $\omega^2 = S_l^2$

$$r_{1,l} = \frac{\sqrt{l(l+1)}c_0}{\omega}$$



$r_{1,l}$ increases as l increases

=> larger degree modes have shallower cavities

For fixed l: $r_{1,l}$ increases as ω increases

=> higher frequency modes propagate deeper, for fixed degree

Trapping of oscillations

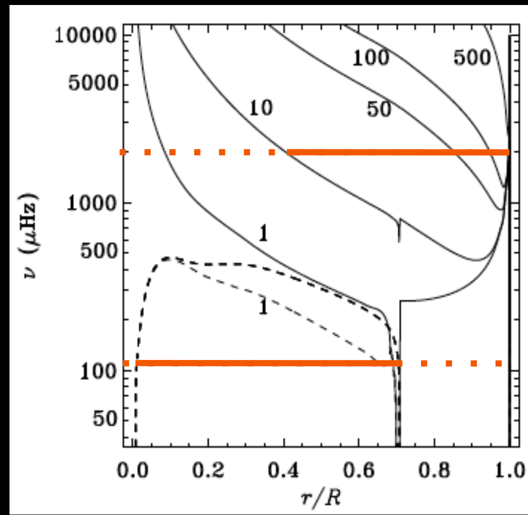
A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- High frequency modes $\omega^2 >> N_0^2$

Near the surface $$k_r^2 \approx \frac{\omega^2-\omega_c^2}{c_0^{\,2}}$$

Upper turning point



Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

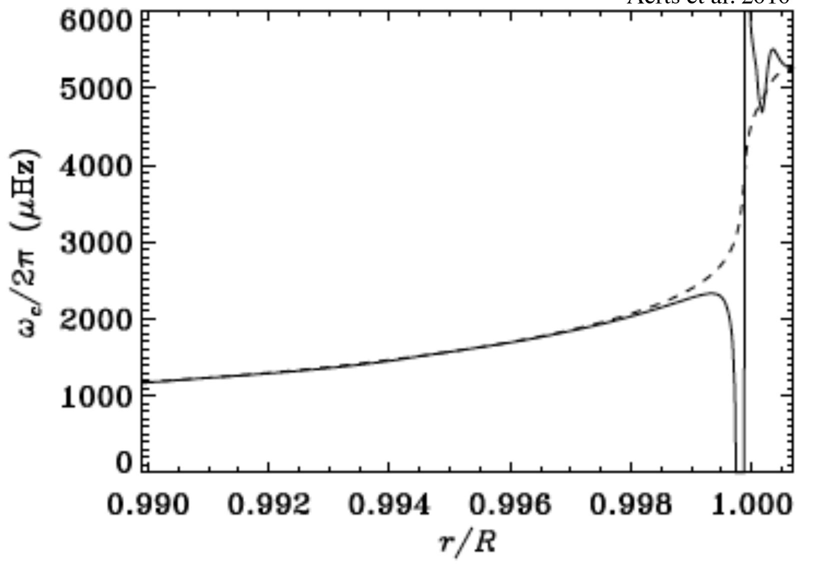
- High frequency modes $\omega^2 >> N_0^2$

Near the surface

$$k_r^2 \approx \frac{\omega^2-\omega_c^2}{c_0^{\,2}}$$

Upper turning point $\omega^2 = \omega_c^2$

Aerts et al. 2010



Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

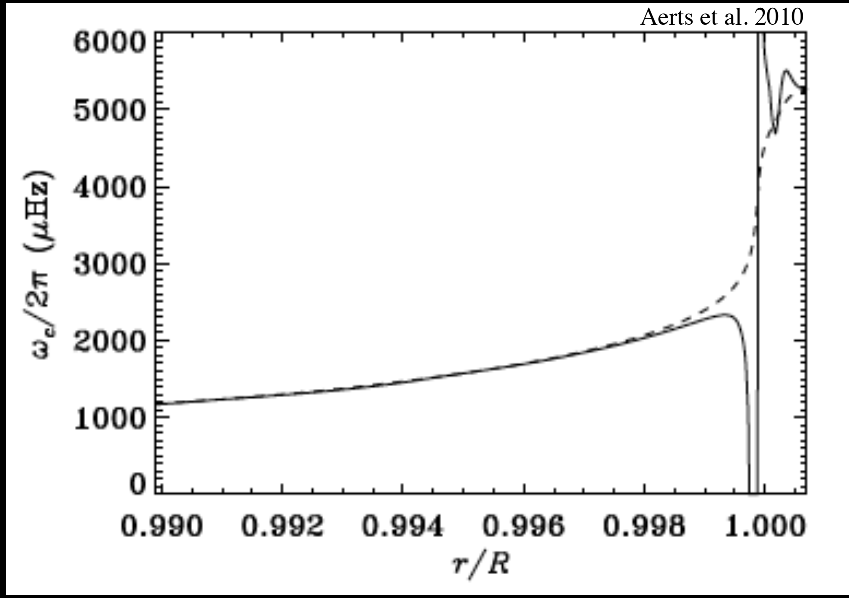
- High frequency modes $\omega^2 >> N_0^2$

Near the surface

$$k_r^2 \approx \frac{\omega^2-\omega_c^2}{c_0^{\,2}}$$

Upper turning point $\omega^2 = \omega_c^2$

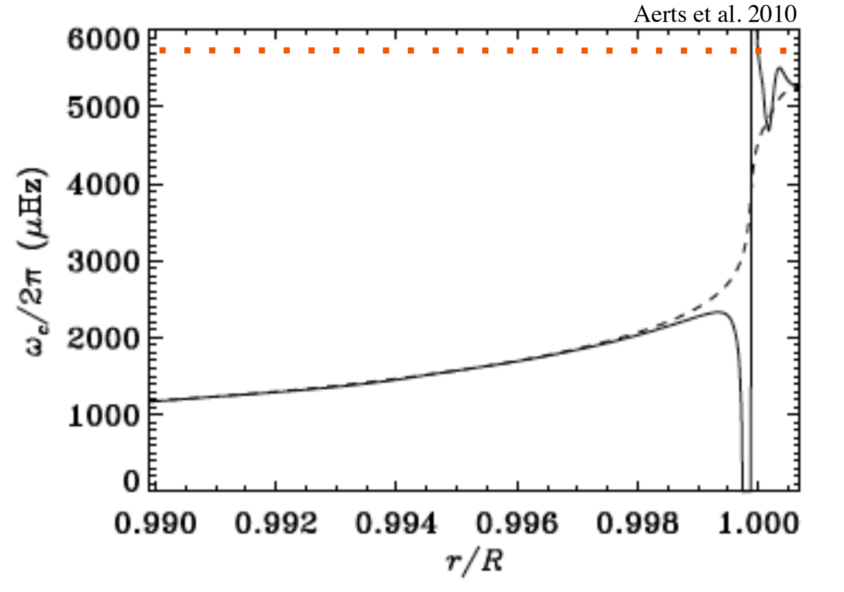
$$\omega \approx \frac{c_0}{2H}\left[1-2\frac{dH}{dr}\right]$$



Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$



- High frequency modes $\omega^2 >> N_0^2$

Near the surface

$$k_r^2 \approx \frac{\omega^2-\omega_c^2}{c_0^{\,2}}$$

Upper turning point $\omega^2 = \omega_c^2$

$$\omega \approx \frac{c_0}{2H}\left[1-2\frac{dH}{dr}\right]$$

Trapping of modes occurs up to ~ 5.3 mHz in the sun
… but partial reflection occurs at even higher frequencies

Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^2}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- High frequency modes $\omega^2 >> N_0^2$

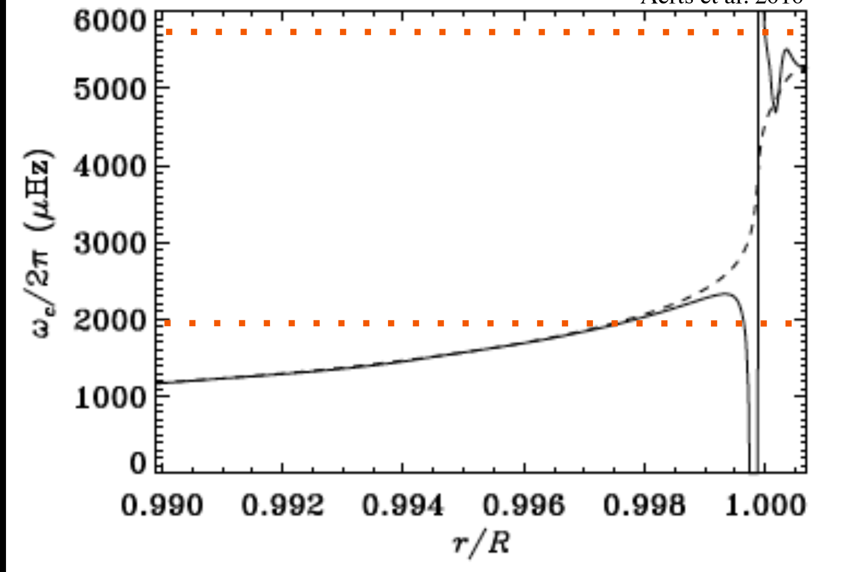
Near the surface

$$k_r^2 \approx \frac{\omega^2-\omega_c^2}{c_0^2}$$

Upper turning point $\omega^2 = \omega_c^2$

$$\omega \approx \frac{c_0}{2H}\left[1-2\frac{dH}{dr}\right]$$

Aerts et al. 2010



Trapping of modes occurs up to ~ 5.3 mHz in the sun
… but partial reflection occurs at even higher frequencies

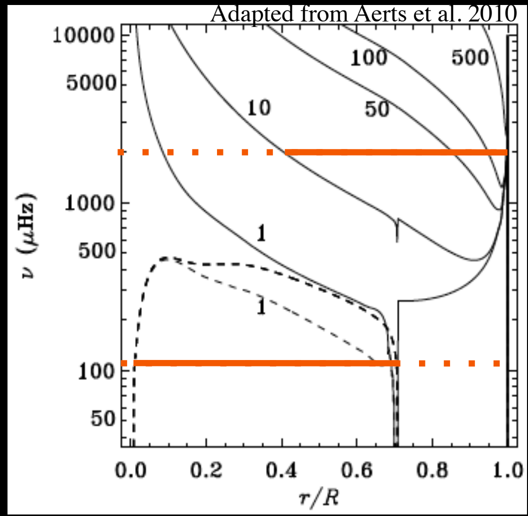
Modes with frequencies lower than ~2 mHz in the sun are reflected below the photosphere
=> not so affected by the details of the outermost layers

Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^{\ 2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- Low frequency modes $\omega^2 << S_l^2$



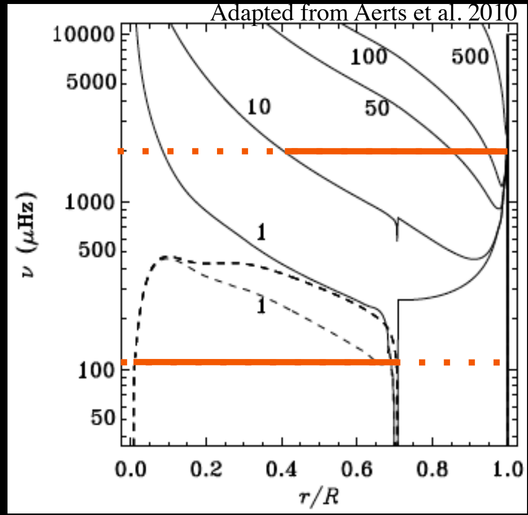
Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^{\ 2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- Low frequency modes $\omega^2 << S_l^2$

$$k_r^2 \approx \frac{S_l^2}{c_0^{\ 2}}\left[\frac{N_0^{\ 2}}{\omega^2}-1+\frac{\omega^2}{S_l^2}-\frac{\omega_c^2}{S_l^2}\right]$$



Trapping of oscillations

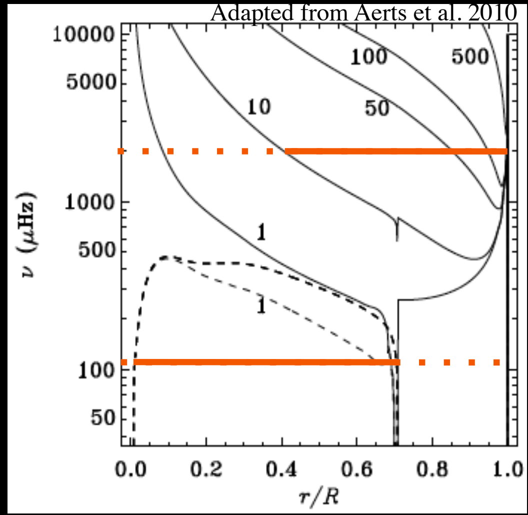
A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

➢ Low frequency modes $\omega^2 << S_l^2$

$$k_r^2 \approx \frac{S_l^2}{c_0^{\,2}}\left[\frac{N_0^{\,2}}{\omega^2}-1+\frac{\omega^2}{S_l^2}-\frac{\omega_c^2}{S_l^2}\right] \approx \frac{l(l+1)}{r^2}\left[N_0^{\,2}-\omega^2\right]\frac{1}{\omega^2}$$

$k_h^{\,2}$



Trapping of oscillations

A closer look at the two families of solutions

$$k_r^2 = \frac{1}{c_0{}^2}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

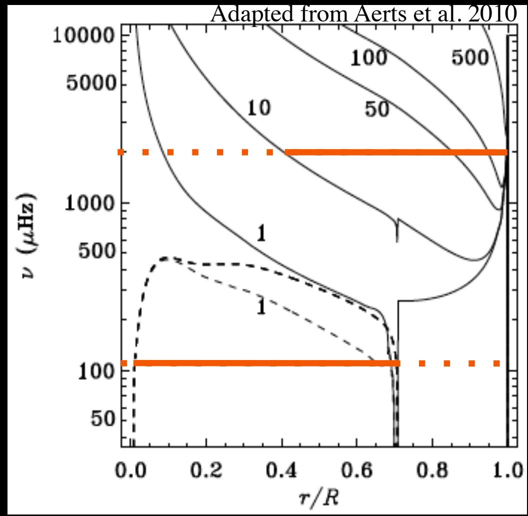
- Low frequency modes $\omega^2 << S_l^2$

$$k_r^2 \approx \frac{S_l^2}{c_0{}^2}\left[\frac{N_0{}^2}{\omega^2}-1+\frac{\omega^2}{S_l^2}-\frac{\omega_c^2}{S_l^2}\right] \approx \frac{l(l+1)}{r^2}\left[N_0{}^2-\omega^2\right]\frac{1}{\omega^2}$$

$k_h{}^2$

$$\omega^2 \approx \frac{N_0^2}{1+\frac{k_r^2}{k_h^2}}$$

Dispersion relation for gravity wave.



Trapping of oscillations

A closer look at the two families of solutions

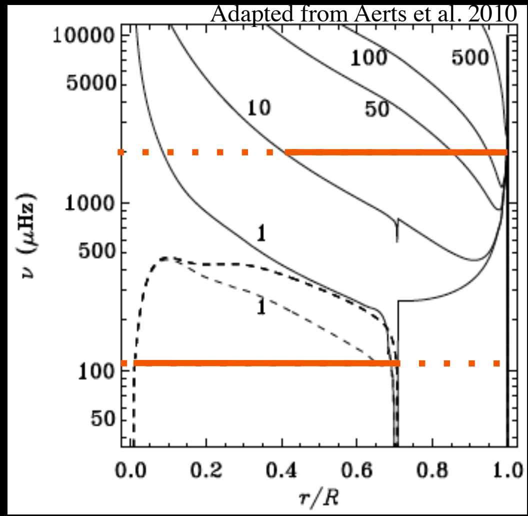
$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- Low frequency modes $\omega^2 << S_l^2$

$$k_r^2 \approx \frac{S_l^2}{c_0^{\,2}}\left[\frac{N_0^{\,2}}{\omega^2}-1+\frac{\omega^2}{S_l^2}-\frac{\omega_c^2}{S_l^2}\right] \approx \frac{l(l+1)}{r^2}\left[N_0^{\,2}-\omega^2\right]\frac{1}{\omega^2}$$

$k_h^{\,2}$

$$\omega^2 \approx \frac{N_0^2}{1+\frac{k_r^2}{k_h^2}}$$



Dispersion relation for gravity wave.

$\omega < N_0$

ω decreases as k_r increases

=> $|n|$ increases as frequency decreases

Trapping of oscillations

A closer look at the two families of solutions

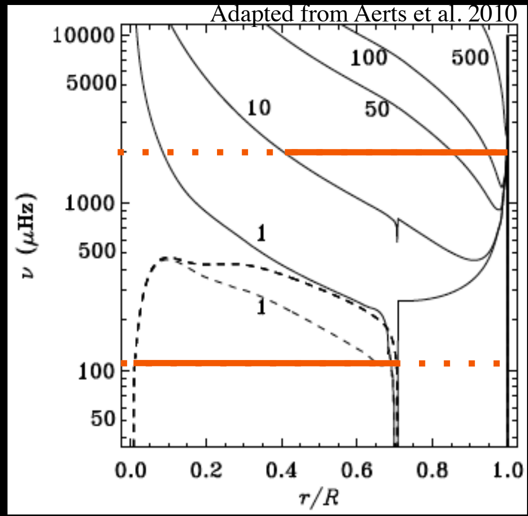
$$k_r^2 = \frac{1}{c_0^2}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- Low frequency modes $\omega^2 << S_l^2$

$$k_r^2 \approx \frac{S_l^2}{c_0^2}\left[\frac{N_0^2}{\omega^2}-1+\frac{\omega^2}{S_l^2}-\frac{\omega_c^2}{S_l^2}\right] \approx \frac{l(l+1)}{r^2}\left[N_0^2-\omega^2\right]\frac{1}{\omega^2}$$

k_h^2

$$\omega^2 \approx \frac{N_0^2}{1+\frac{k_r^2}{k_h^2}}$$



Dispersion relation for gravity wave.

Smaller k_r/k_h => Larger λ_r/λ_h => larger ω

=> larger frequencies for "needle-like" motion

The frequency of a gravity wave is always smaller that N_0

Trapping of oscillations

A closer look at the two families of solutions

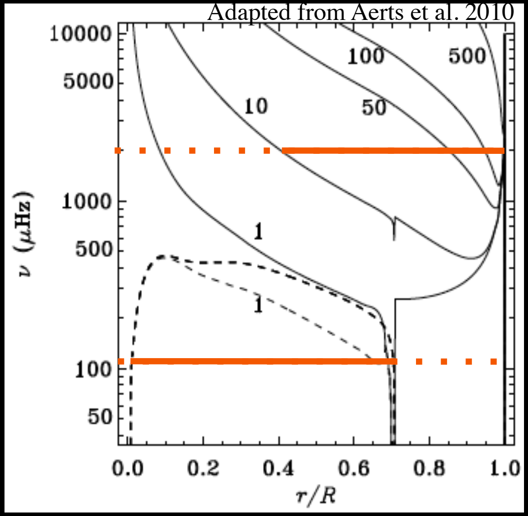
$$k_r^2 = \frac{1}{c_0^{\,2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- Low frequency modes $\omega^2 << S_l^2$

$$k_r^2 \approx \frac{S_l^2}{c_0^{\,2}}\left[\frac{N_0^{\,2}}{\omega^2}-1+\frac{\omega^2}{S_l^2}-\frac{\omega_c^2}{S_l^2}\right] \approx \frac{l(l+1)}{r^2}\left[N_0^{\,2}-\omega^2\right]\frac{1}{\omega^2}$$

$k_h^{\,2}$

Turning points



Trapping of oscillations

A closer look at the two families of solutions

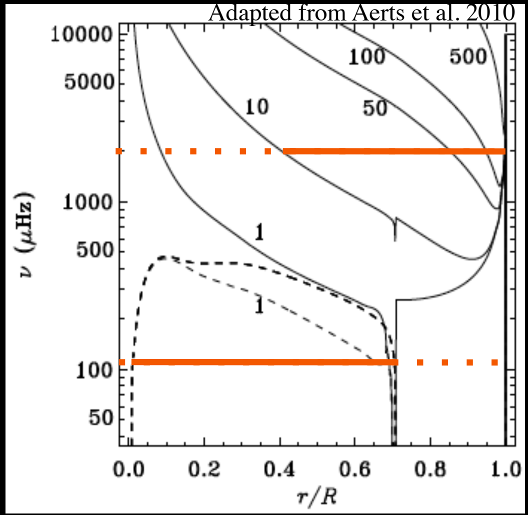
$$k_r^2 = \frac{1}{c_0^{\ 2}}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- Low frequency modes $\omega^2 << S_l^2$

$$k_r^2 \approx \frac{S_l^2}{c_0^{\ 2}}\left[\frac{N_0^{\ 2}}{\omega^2}-1+\frac{\omega^2}{S_l^2}-\frac{\omega_c^2}{S_l^2}\right] \approx \frac{l(l+1)}{r^2}\left[N_0^{\ 2}-\omega^2\right]\frac{1}{\omega^2}$$

$k_h^{\ 2}$

Turning points $\omega^2 = N_0^2$



Gravity waves propagate only in convectively stable regions!

Trapping of oscillations

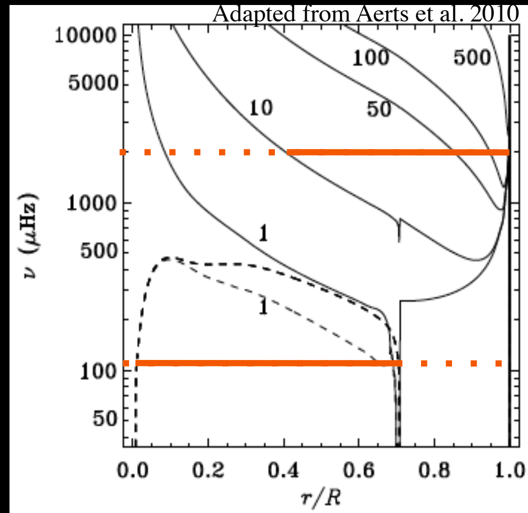
A closer look at the two families of solutions

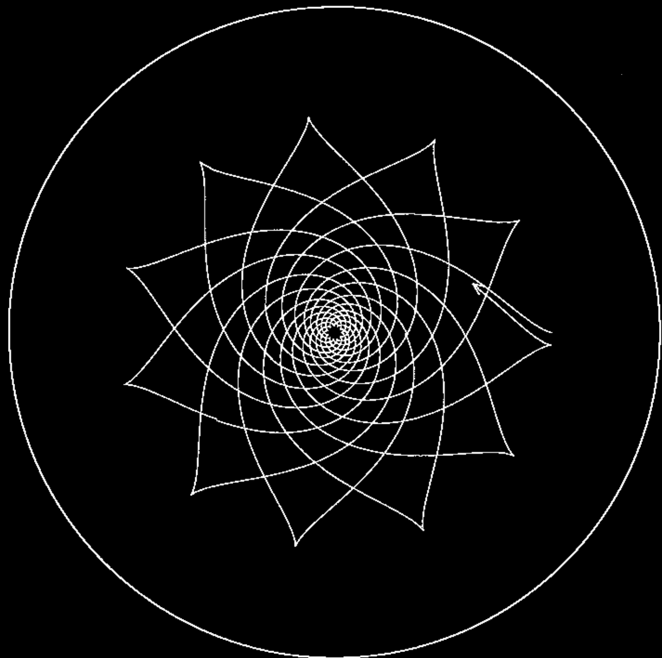
$$k_r^2 = \frac{1}{c_0^2}\left[S_l^2\left(\frac{N_0^2}{\omega^2}-1\right)+\omega^2-\omega_c^2\right]$$

- Low frequency modes $\omega^2 << S_l^2$

$$k_r^2 \approx \frac{S_l^2}{c_0^2}\left[\frac{N_0^2}{\omega^2}-1+\frac{\omega^2}{S_l^2}-\frac{\omega_c^2}{S_l^2}\right] \approx \underbrace{\frac{l(l+1)}{r^2}}_{k_h^2}\left[N_0^2-\omega^2\right]\frac{1}{\omega^2}$$

Turning points $\omega^2 = N_0^2$





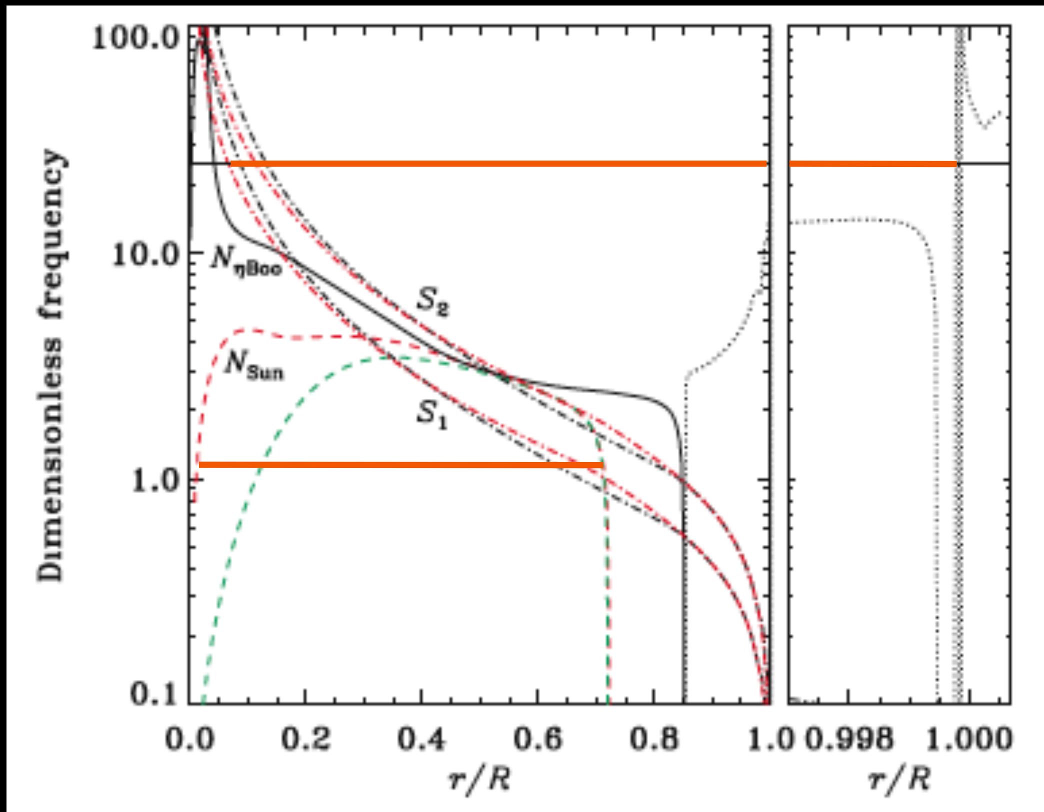
The case of an evolved star

Trapping of oscillations

The case of an evolved star

- Propagation diagram for the sun and a subgiant star

Cunha et al. 2007

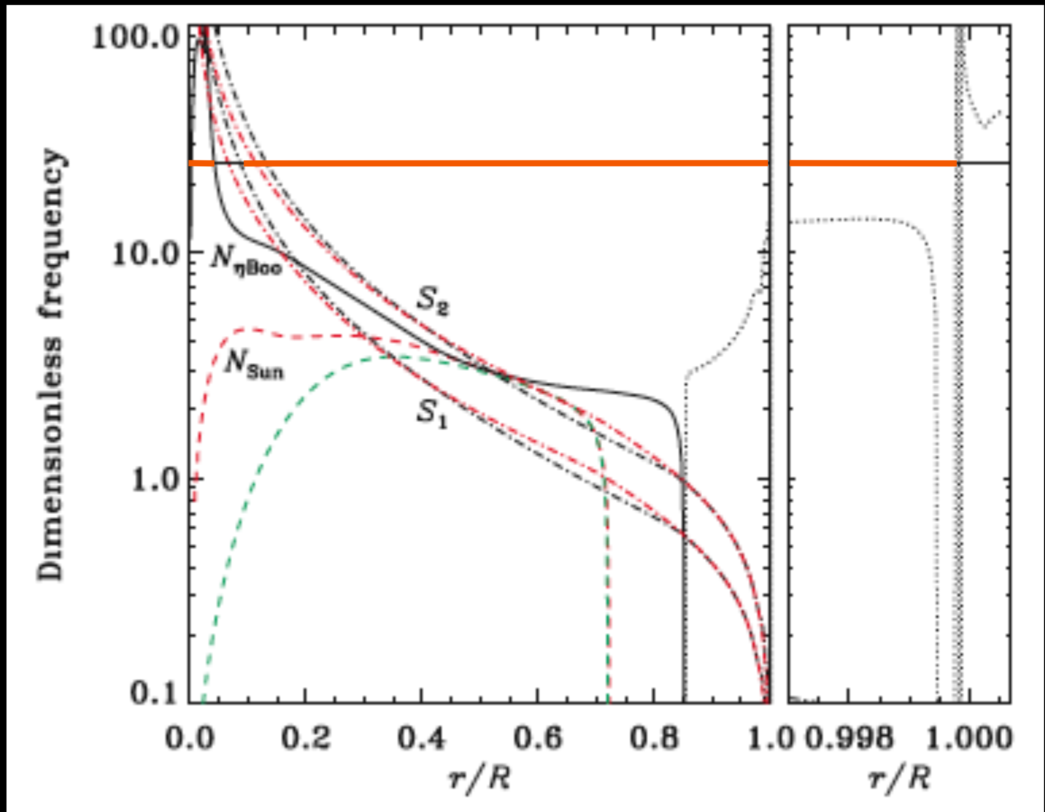


Trapping of oscillations

The case of an evolved star

- Propagation diagram for the sun and a subgiant star

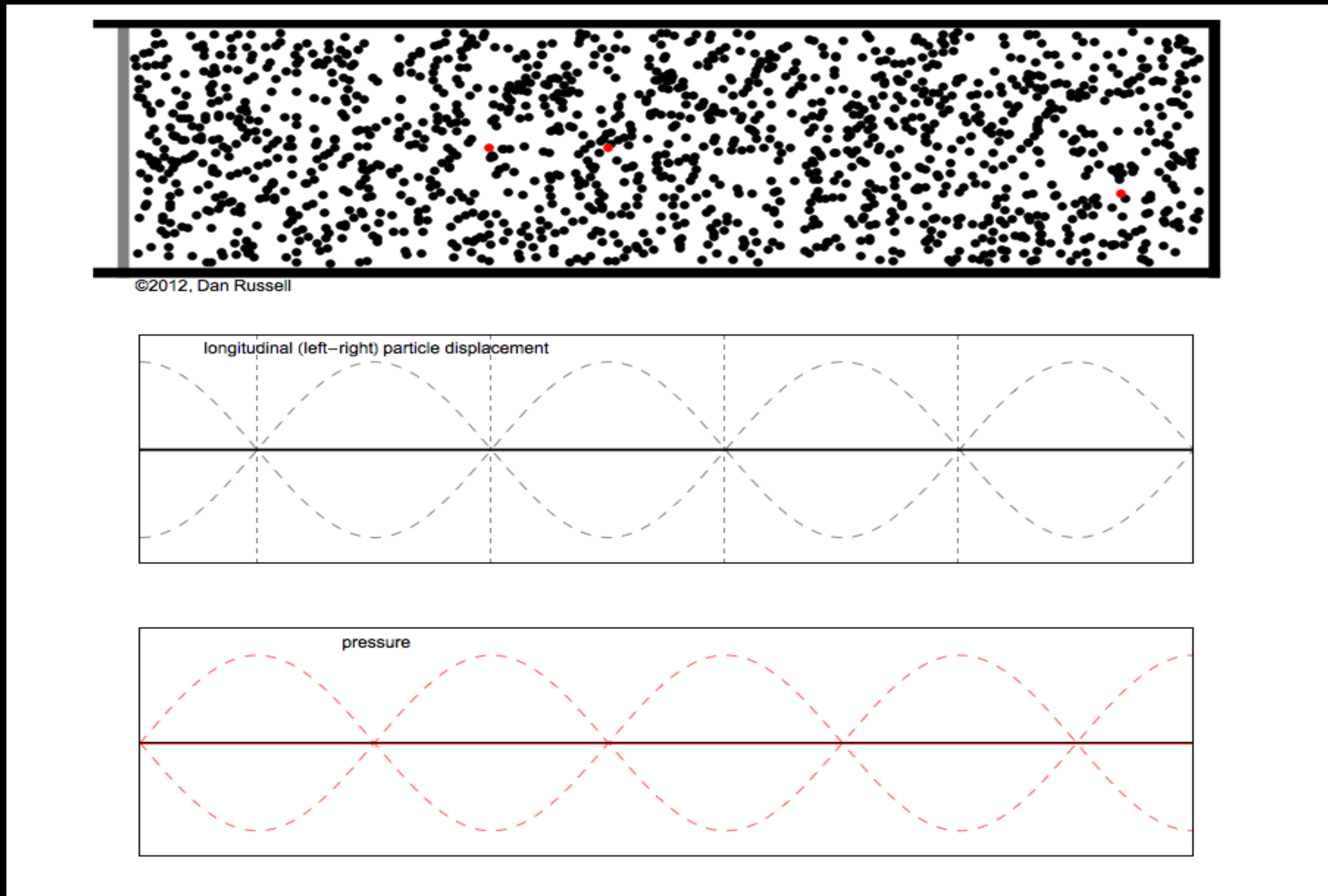
Cunha et al. 2007



Acoustic and internal gravity waves

Acoustic and gravity waves

Acoustic wave



Acoustic and gravity waves

Internal gravity wave

http://www.phys.ocean.dal.ca/programs/doubdiff/pics/iw1.mpeg

Acoustic and gravity waves

Summary

Acoustic waves

- Maintained by gradient of pressure fluctuation;
- Radial or non-radial;
- Propagate in convectively stable or non-stable regions

Internal gravity waves

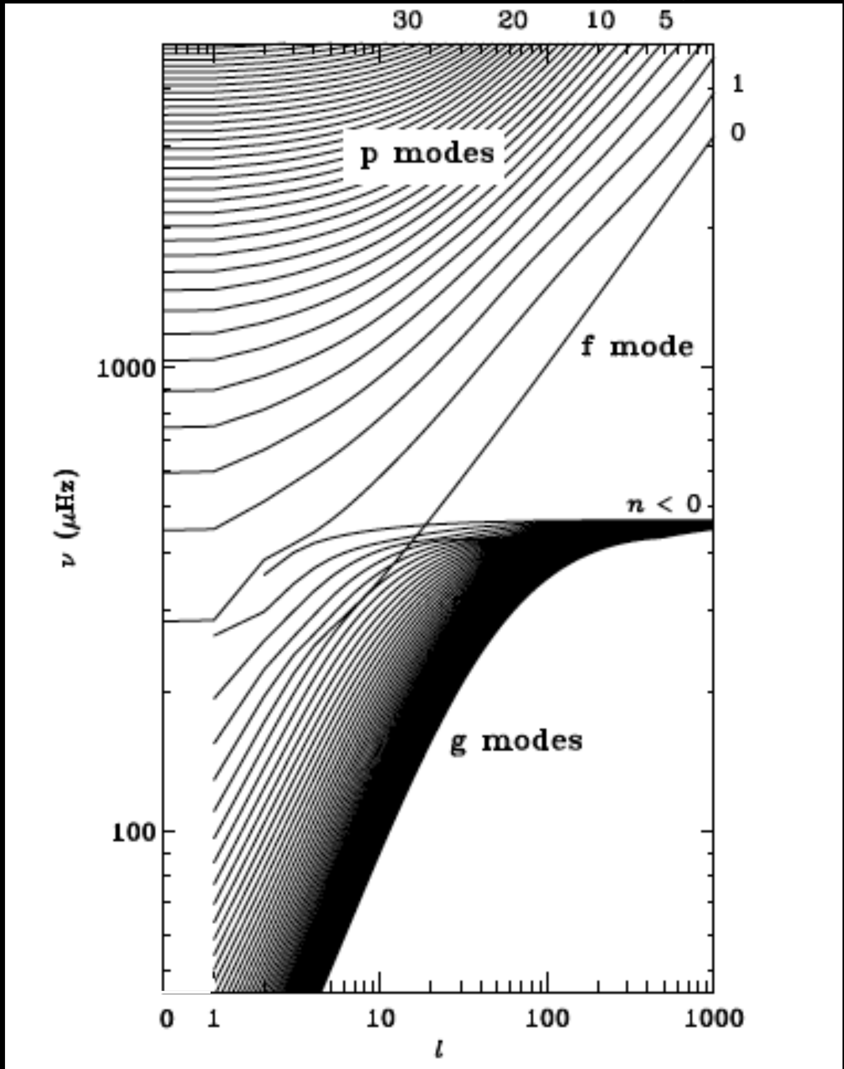
- Maintained by gravity acting on density fluctuation;
- Always non-radial;
- Propagate in convectively stable regions only

Numerical solutions

Numerical results

Eigenfrequencies

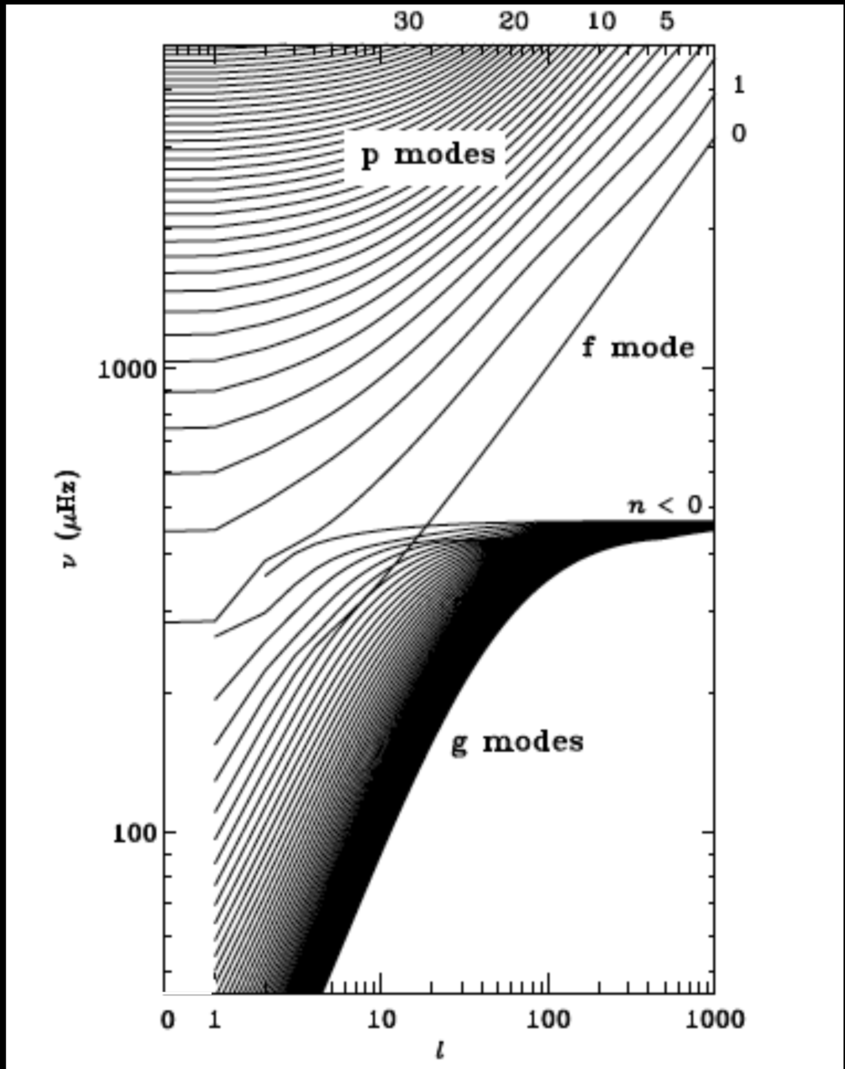
Aerts et al. 2010

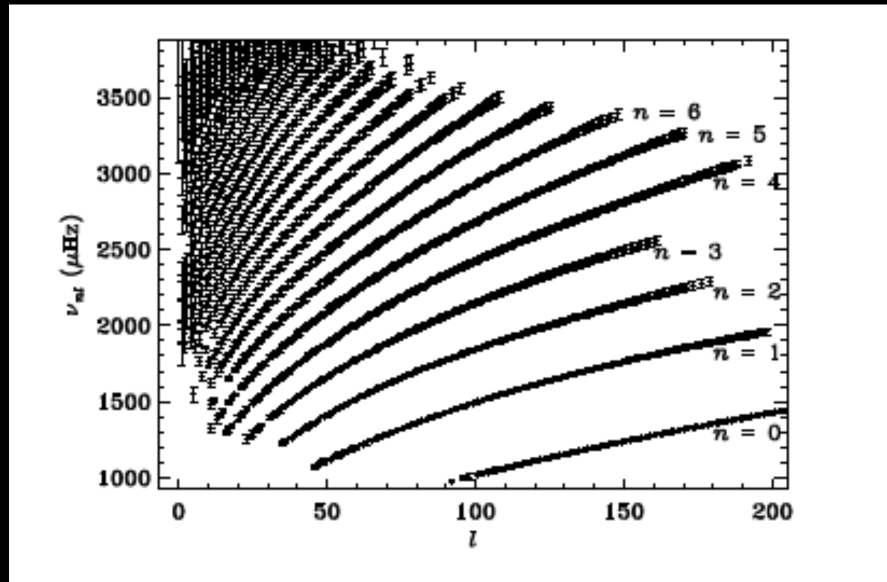


Numerical results

Eigenfrequencies

Aerts et al. 2010



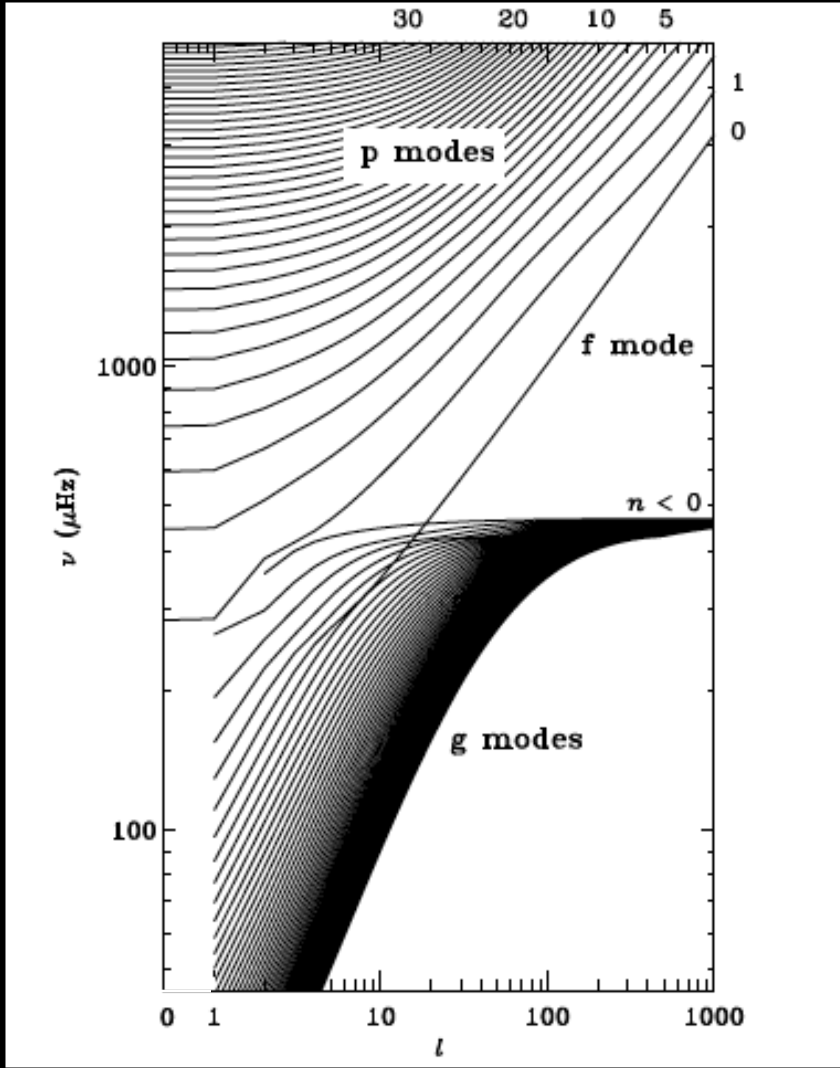


MDI observations

Numerical results

Eigenfrequencies

Aerts et al. 2010



Acoustic modes: $n > 0$

Gravity modes: $n < 0$

Numerical results

Eigenfrequencies

Remember

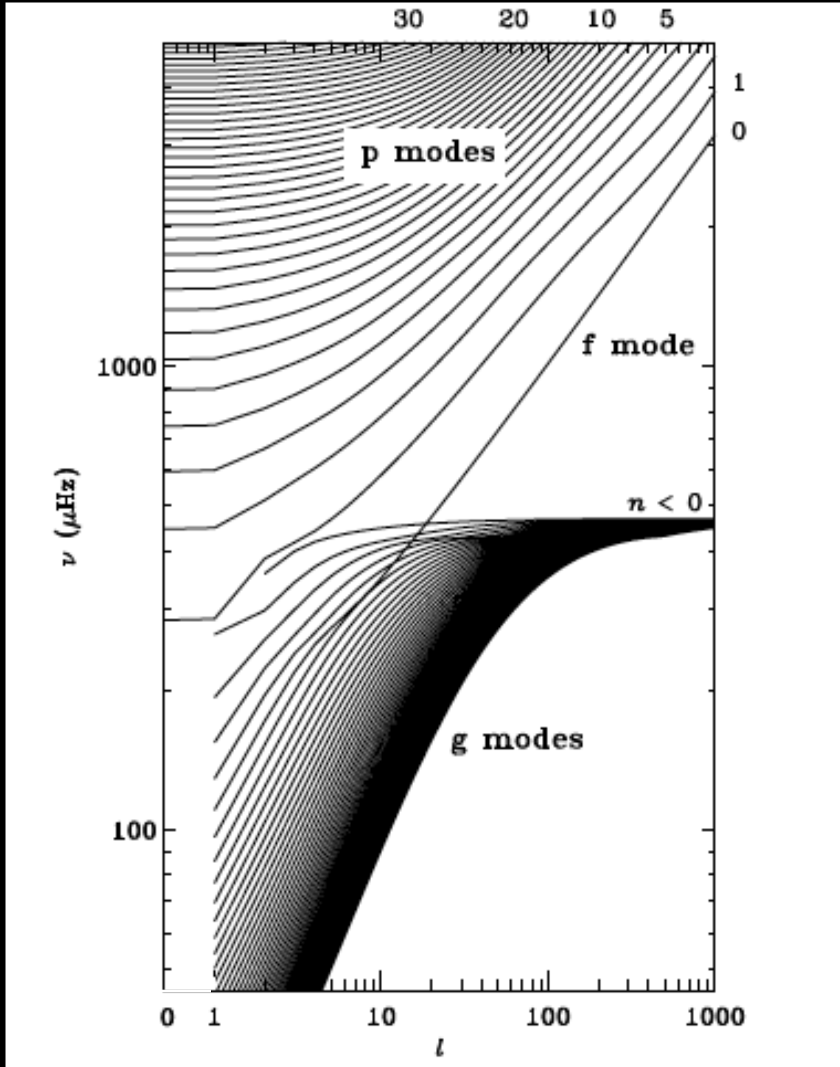
Acoustic waves

$$\omega \approx c_0 k$$

$$\omega^2 \approx \frac{N_0^2}{1 + \frac{k_r^2}{k_h^2}}$$

Gravity waves

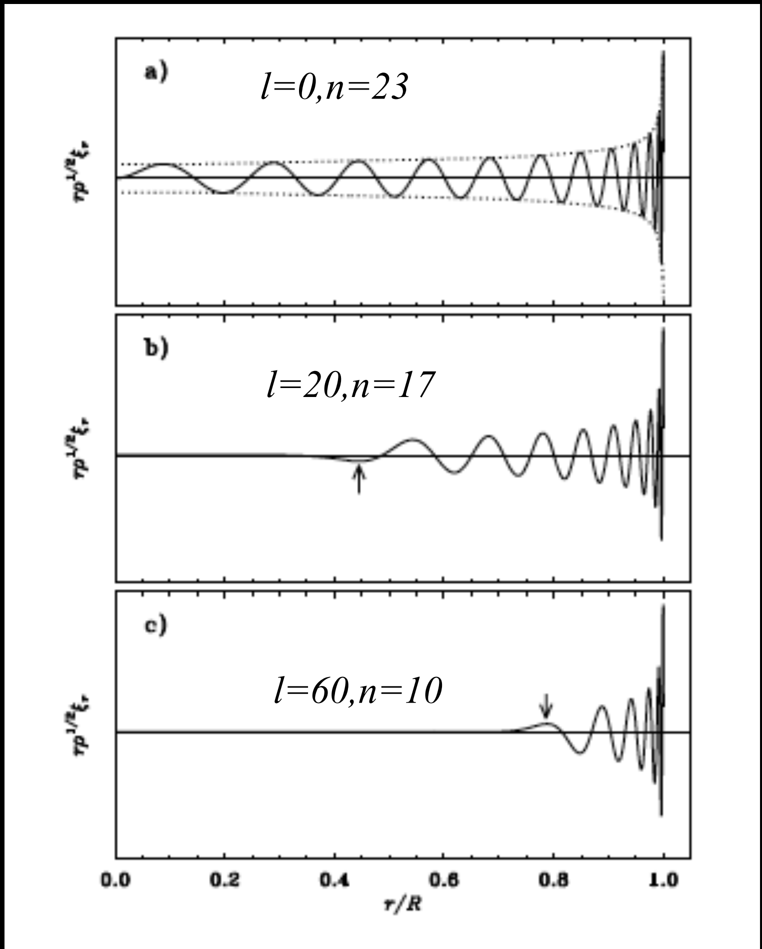
Aerts et al. 2010

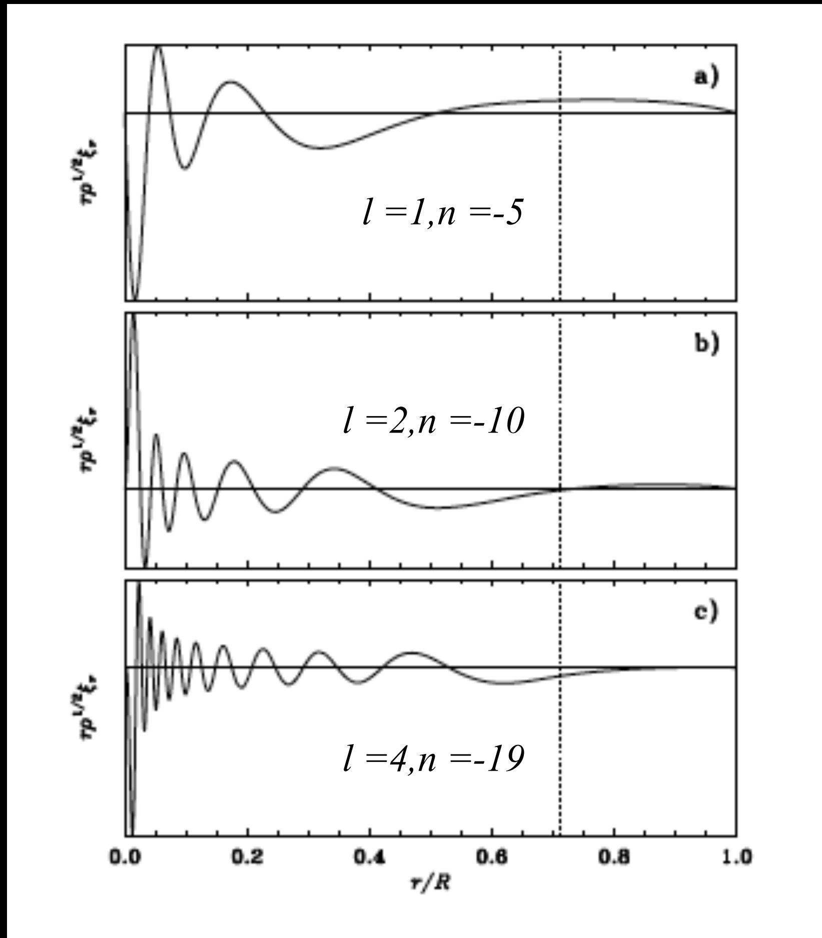


Numerical results

Eigenfunctions

Aerts et al. 2010

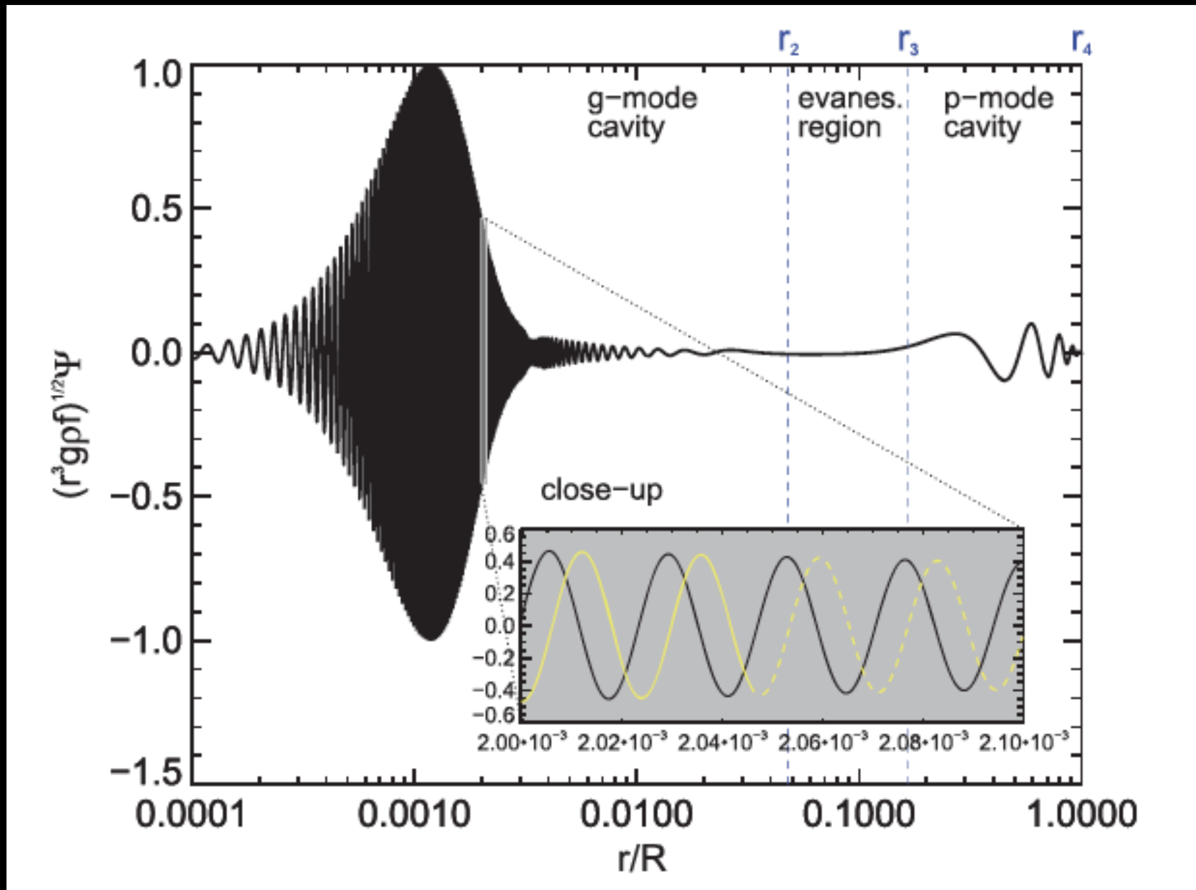




Numerical results

Eigenfunctions

Cunha et al. 2015



A number of important things that were left out

- The actual asymptotic analysis:

  => analytical solutions for the eigenfunctions and eigenfrequencies

- Frequency combinations (large separation, small separations, ratios, etc)

- Inference methodologies (forward modelling, inverse modelling, glitches, etc)

- Deviations from spherical symmetry (rotation, magnetic effects, application of the variational principle)

- Mode excitation (stochastic, coherent)

- etc…

Asymptotic analysis

Linear, adiabatic oscillations in the Cowling approximation.

High n, low l, acoustic oscillations:

$$\nu_{nl} \approx \left(n + \frac{l}{2} + \alpha \right) \Delta\nu_0 + higher\ order\ terms$$

where $$\Delta\nu_0 = \left(2 \int_0^R \frac{dr}{c} \right)^{-1}$$

- $\Delta\nu_0$ prop $(M/R^3)^{1/2}$
- α function of ν and is due to surface effects
- Note: $\nu = \omega/2\pi$

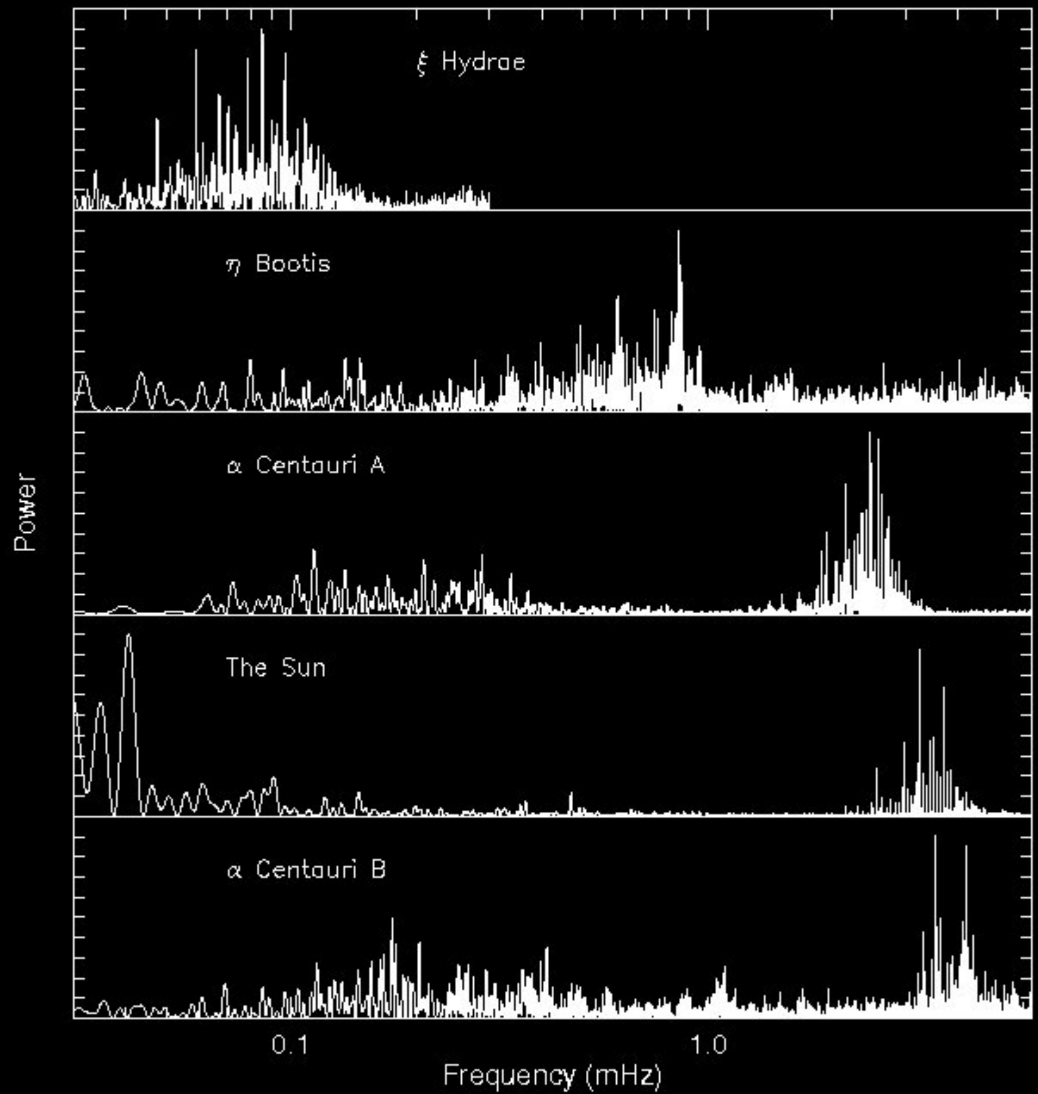
Asymptotic analysis

Adiabatic oscillations in the Cowling approximation.

High *n*, low *l*, acoustic oscillations:

$$\nu_{nl} \approx \left(n + \frac{l}{2} + \alpha \right) \Delta\nu_0 + \ldots$$

$\Delta\nu_0$ prop $(M/R^3)^{1/2}$



Asymptotic analysis

Large separations $\Delta\nu_{nl}$

$$\nu_{nl} \approx \left(n + \frac{l}{2} + \alpha \right) \Delta\nu_0 + higher\ order\ terms$$

Asymptotic analysis

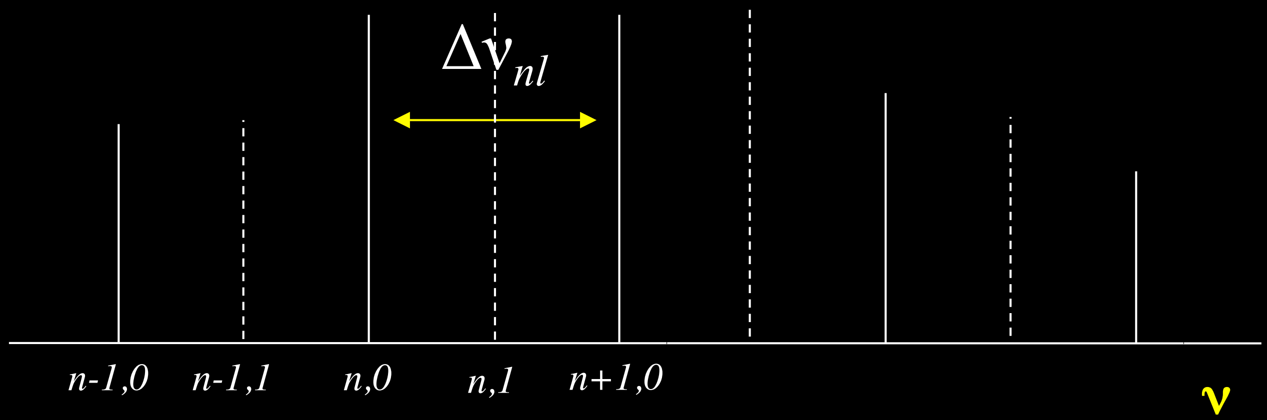
Large separations $\Delta\nu_{nl}$

$$\nu_{nl} \approx \left(n + \frac{l}{2} + \alpha \right) \Delta\nu_0 + higher\ order\ terms$$

$$\Delta\nu_{nl} = \nu_{n+1,l} - \nu_{n,l} \approx \Delta\nu_0$$

$\alpha\ (M/R^3)^{1/2}$

Schematic
Power
Spectrum



Asymptotic analysis

Adiabatic oscillations in the Cowling approximation.

High n, low l, acoustic oscillations:

$$\nu_{nl} \approx \left(n + \frac{l}{2} + \alpha \right) \Delta\nu_0 - \left[Al(l+1) - \delta \right] \frac{\Delta\nu_0}{\nu_{nl}} + \ldots$$

where $$A = \frac{1}{4\pi^2 \Delta\nu_0} \left[\frac{c(R)}{R} - \int_0^R \frac{dc}{r} \right]$$

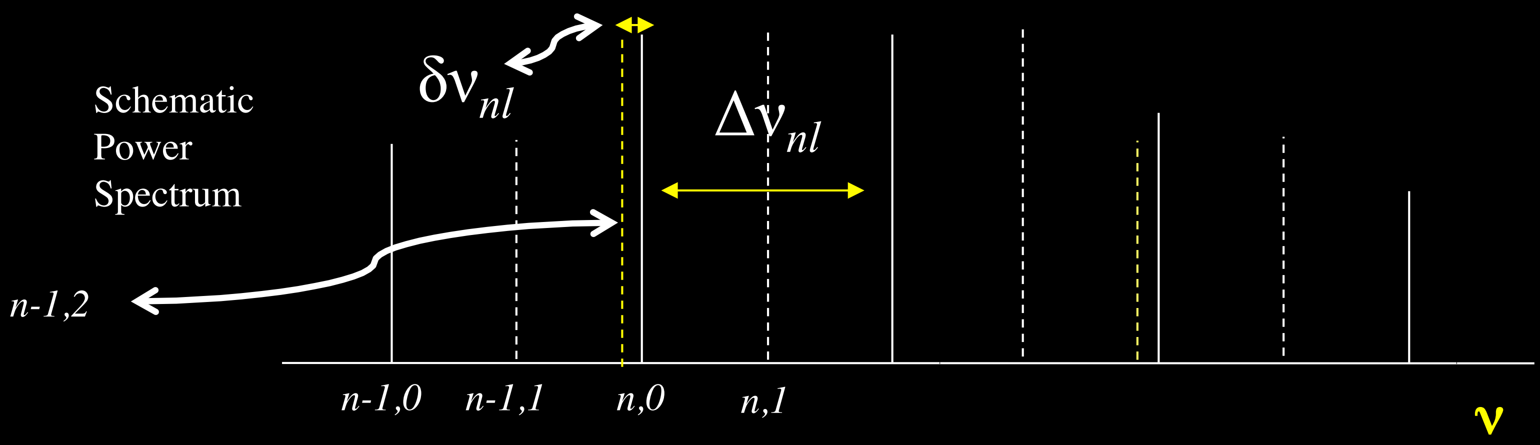
Asymptotic analysis

small separations $\delta\nu_{nl}$

$$\nu_{nl} \approx \left(n + \frac{l}{2} + \alpha\right)\Delta\nu_0 - \left[Al(l+1) - \delta\right]\frac{\Delta\nu_0}{\nu_{nl}} + \dots$$

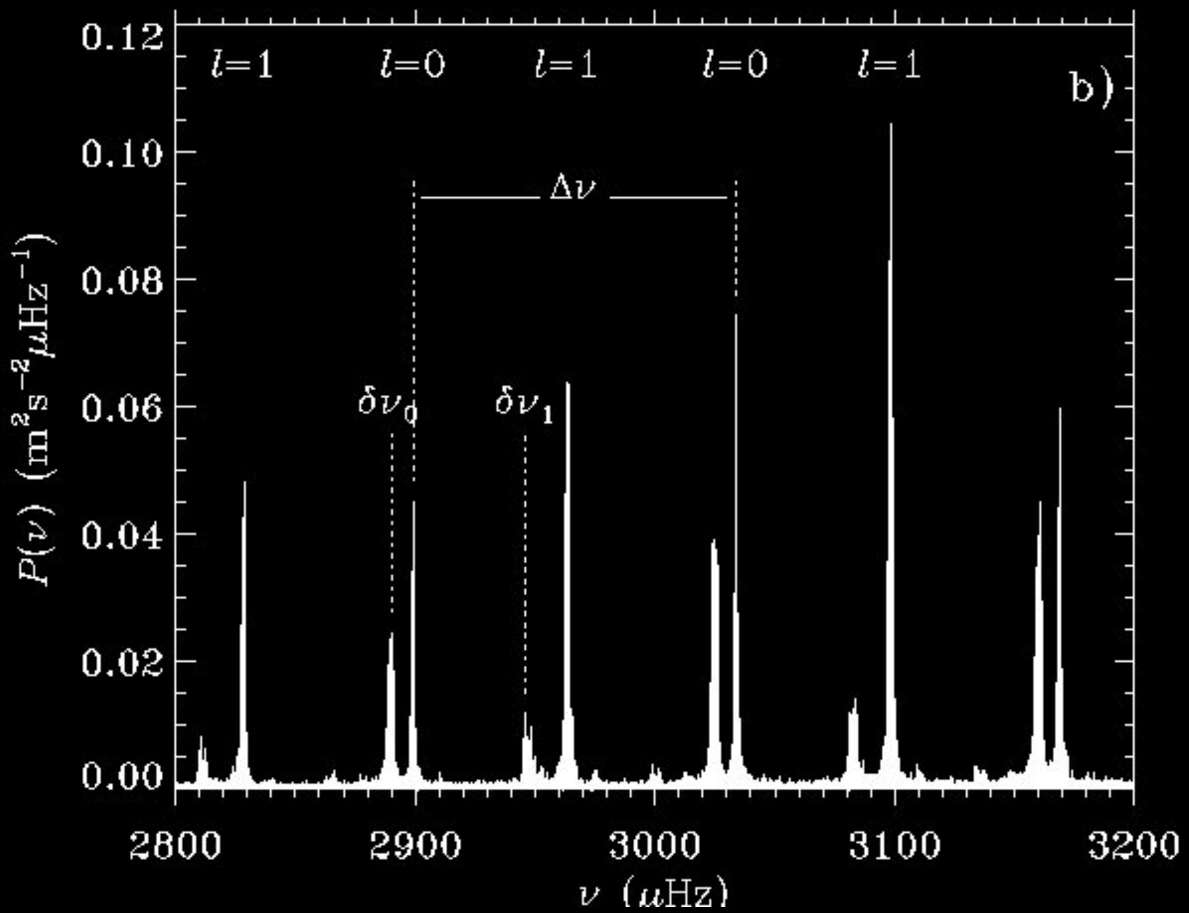
where $$A = \frac{1}{4\pi^2 \Delta\nu_0}\left[\frac{c(R)}{R} - \int_0^R \frac{dc}{r}\right]$$

$$\delta\nu_{nl} = \nu_{n,l} - \nu_{n-1,l+2} \approx -(4l+6)\frac{\Delta\nu_0}{4\pi^2 \nu_{n,l}}\int_0^R \frac{dc}{r}$$



Asymptotic analysis

Sun as a star



Enjoy the rest of the school and have a great stay at Azores!

THE END