Stellar Inversion Techniques

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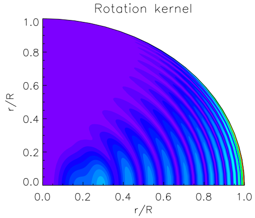


My CV

- 2006: obtained PhD in Toulouse
- 2006-2015: 4 postdocs
  - Sheffield: supervisor = M. J. Thompson
  - Meudon (near Paris)
  - Liège: supervised G. Buldgen
  - Birmingham
- 2015-???: associate-astronomer in Meudon

Forward and inverse problems

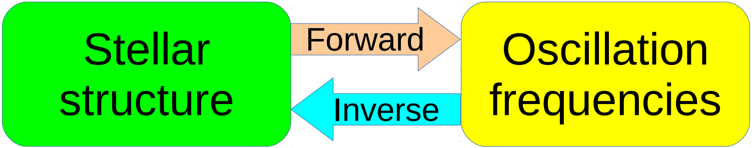
Forward problem

- start from a set of physical causes, and deduce the results
- examples:
  - deduce earth's gravity field from the distribution of matter in the earth
  - calculate the distorted image of an object through a given optical system

Inverse problem

- deduces causes from a given set of results (which are typically observations)
- examples:
  - deduce the distribution of matter in the earth based on the measured gravitational field
  - removing optical distortion to find the true geometrical shape of an object

Helio- and asteroseismology



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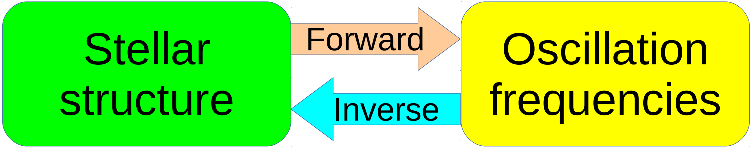
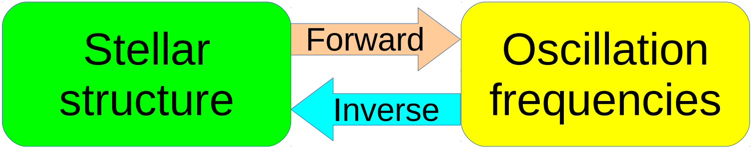


Forward problem

- requires stellar evolution code or model + stellar oscillations code
- in general, non-linear relation between structure and oscillation frequencies
- see "Theory of Stellar Oscillations" lecture by M. Cunha (CA2)

Helio- and asteroseismology



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Inverse problem

- in general, non-linear problem
- variety of different approaches

Different approaches for solving the inverse problem

- according to Gough (1985) there are 3 ways of inverting helioseismic data:
  1. repeated "execution of the forward problem" (*i.e.* search in parameter space, or "forward modelling")
  2. analytical methods (asymptotic methods, glitch fitting)
  3. formal inversion techniques

Different approaches for solving the inverse problem

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  1. repeated "execution of the forward problem" (*i.e.* search in parameter space, or "forward modelling")
  2. analytical methods (asymptotic methods, glitch fitting)
  3. formal inversion techniques
- this also applies to asteroseismology. However:
  - greater uncertainties on "classical parameters" ($T_{\rm eff}$, [Fe/H], L, $v \sin i$ ...)
  - fewer number of available frequencies

Comparison between different approaches

Forward modelling

- **description**: search for optimal model in a restricted parameter space
- **advantages**: simplicity, physically coherent models
- see AIMS tutorial by M. Lund and D. Reese (TA2)

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Inversion techniques

- **description**: adjust the structure of a reference model so as to match the observed frequencies
- **advantages**: extracts more information from frequencies, open to new physics

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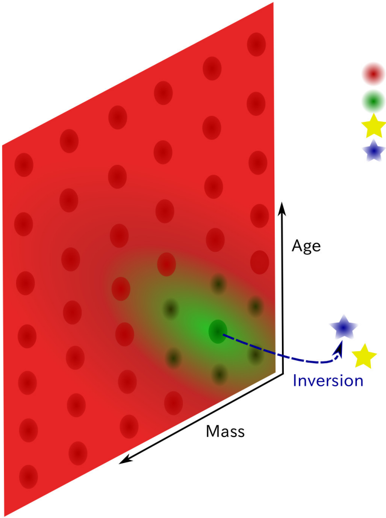
Comparison

- the two approaches are in fact complementary:
  - the direct approach can provide an initial model for an inverse method

Comparison between different approaches



- A model from the grid (poor fit)
- Optimal model from the grid
- The star
- Inverted stellar model

(image courtesy of G. Buldgen)

1. Introduction
2. The forward problem
   - Adiabatic pulsation equations
   - Symmetry of the pulsation equations
   - Kernels
3. Inversion techniques
   - Rotation inversions
   - Structural inversions
   - Integrated quantities
   - Non-linear inversion methods
4. Conclusion

Adiabatic pulsation equations

- Euler's equation

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = -\frac{\vec{\nabla} p'}{\rho_0} + \frac{\rho' \vec{g}_0}{\rho_0} - \vec{\nabla}\Psi'$$

- continuity equation

$$\rho' + \vec{\xi} \cdot \vec{\nabla}\rho_0 + \rho_0 \vec{\nabla} \cdot \vec{\xi} = 0$$

- adiabatic relation

$$p' - c_0^2 \rho' = \vec{\xi} \cdot \left(c_0^2 \vec{\nabla}\rho_0 - \vec{\nabla} P_0 \right)$$

- Poisson's equation

$$\Delta\Psi' = 4\pi G \rho'$$

- variables with a "0" subscript are equilibrium quantities
- variables with a prime ($'$) are Eulerian perturbations
- $\vec{\xi}$ denotes the Lagrangian displacement

Adiabatic pulsation equations

- the last three equations along with appropriate boundary conditions, enable us to express ρ', p', and Ψ' in terms of $\vec{\xi}$:

$$
\begin{aligned}
\rho' &= -\vec{\xi}\cdot\vec{\nabla}\rho_0 - \rho_0\vec{\nabla}\cdot\vec{\xi} \\
p' &= -c_0^2\rho_0\vec{\nabla}\cdot\vec{\xi} - \xi\cdot\vec{\nabla}P_0 \\
\Psi'(\vec{r}) &= -\iiint_V \frac{G\rho'(\vec{r}')}{\|\vec{r}-\vec{r}'\|}\mathrm{d}V
\end{aligned}
$$

- The above equations can also be applied, even if $\vec{\xi}$ is not an eigenmode, thereby leading to a set of variables $(\vec{\xi}, \rho', p', \Psi')$ which we shall call a "partial solution".
- To obtain a full solution, Euler's equation (along with suitable boundary conditions) still needs to be applied.

Adiabatic pulsation equations

- substituting the above expressions for p', ρ' and Ψ' into Euler's equation then leads to the following schematic equation:

$$\omega^2\vec{\xi} = \mathcal{F}(\vec{\xi})$$

where:

  - $\mathcal{F}$ is an integro-differential operator that depends on stellar structure
  - we've assumed $\vec{\xi} \propto \exp(-i\omega t)$ (hence, $\frac{\partial\vec{\xi}}{\partial t} \equiv -i\omega\vec{\xi}$)

- this equation is an eigenvalue problem, the solutions of which are known as "eigensolutions". These contain two parts:
  - the eigenvalue, ω^2, *i.e.* the square of the pulsation frequency
  - the eigenfunction or eigenmode, $\vec{\xi}$

Non-linear aspects

- the above equation is non-linear
  - need to linearise it, in order to apply seismic inversions:

$$(\delta\omega^2)\vec{\xi} + \omega^2(\delta\vec{\xi}) = \delta\mathcal{F}(\vec{\xi}) + \mathcal{F}(\delta\vec{\xi})$$

- **beware**: here, "δ" is not a Lagrangian perturbation, but rather a modification of the model and its pulsations

- the above linearised equations would have to be solved numerically which could get complicated fairly quickly
- however, this can be greatly simplified thanks to the fact that the operator $\mathcal{F}$ is *symmetric*

Symmetry of the pulsation equations

A dot product

- in order to define what "*symmetry*" means in this context, we introduce the following dot product:

$$\left\langle \vec{\eta}, \vec{\xi} \right\rangle = \int_V \rho_0 \vec{\eta}^* \cdot \vec{\xi} \mathrm{d}V$$

- where:
  - $\vec{\eta}^*$ is the complex conjugate of $\vec{\eta}$
  - V is the stellar volume
- *NOTE:* this is a complex dot product, hence $\left\langle \vec{\eta}, \vec{\xi} \right\rangle = \left\langle \vec{\xi}, \vec{\eta} \right\rangle^*$

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Definition

- the adiabatic pulsation equations are symmetric with respect to the above dot product:

$$\left\langle \vec{\eta}, \mathcal{F}(\vec{\xi}) \right\rangle = \left\langle \mathcal{F}(\vec{\eta}), \vec{\xi} \right\rangle$$

for any displacement fields $\vec{\eta}$, $\vec{\xi}$

Symmetry of the pulsation equations

Definition

- in order to prove the symmetry of the pulsation equations, we introduce two partial solutions:

$$(\vec{\xi}, p', \Psi') \qquad (\vec{\eta}, \pi', \Phi')$$

- we calculate the following dot product:

$$\left\langle \vec{\eta}, \mathcal{F}(\vec{\xi}) \right\rangle$$

- after various manipulations (integration by parts etc.), this leads to the following formula:

$$\begin{aligned} \left\langle \vec{\eta}, \mathcal{F}(\vec{\xi}) \right\rangle &= \int_V \frac{(\pi')^* p'}{\rho_0 c_0^2} \mathrm{d}V + \int_V \rho_0 N_0^2 \eta_r^* \xi_r \mathrm{d}V \\ &+ \int_S \rho_0 g_0 \eta_r^* \xi_r \mathrm{d}S - \frac{1}{4\pi G} \int_{V_\infty} \vec{\nabla}(\Phi')^* \cdot \vec{\nabla}\Psi' \mathrm{d}V \end{aligned}$$

- the above form is symmetric

Symmetry of the pulsation equations

Consequence 1 - ω^2 is real

- the pulsation equations are written as:

$$\omega^2 \vec{\xi} = \mathcal{F}(\vec{\xi})$$

- taking the dot product with $\vec{\xi}$ on both sides of the equation, and isolating ω^2 leads to:

$$\omega^2 = \frac{\left\langle \vec{\xi}, \mathcal{F}(\vec{\xi}) \right\rangle}{\left\langle \vec{\xi}, \vec{\xi} \right\rangle}$$

- the denominator is real and strictly positive
- the numerator is real. This can be seen by taking its complex conjugate and showing it is equal to the original expression:

$$\left\langle \vec{\xi}, \mathcal{F}(\vec{\xi}) \right\rangle^* = \left\langle \mathcal{F}(\vec{\xi}), \vec{\xi} \right\rangle = \left\langle \vec{\xi}, \mathcal{F}(\vec{\xi}) \right\rangle$$

- *note:* the first equality comes from the definition of the dot product, whereas the second equality comes from the symmetry of $\mathcal{F}$

Symmetry of the pulsation equations

Consequence 2 – orthogonality of the eigenmodes

- eigenmodes with distinct eigenvalues are orthogonal

$$
\begin{aligned}
\omega_1^2 \vec{\xi}_1 &= \mathcal{F}(\vec{\xi}_1) \\
\omega_2^2 \vec{\xi}_2 &= \mathcal{F}(\vec{\xi}_2) \\
\omega_1^2 \left\langle \vec{\xi}_2, \vec{\xi}_1 \right\rangle &= \left\langle \vec{\xi}_2, \mathcal{F}(\vec{\xi}_1) \right\rangle = \left\langle \mathcal{F}(\vec{\xi}_2), \vec{\xi}_1 \right\rangle = \omega_2^2 \left\langle \vec{\xi}_2, \vec{\xi}_1 \right\rangle
\end{aligned}
$$

- this can be re-expressed as:

$$
(\omega_1^2 - \omega_2^2) \left\langle \vec{\xi}_2, \vec{\xi}_1 \right\rangle = 0
$$

- hence, either $\omega_1^2 = \omega_2^2$ or $\left\langle \vec{\xi}_2, \vec{\xi}_1 \right\rangle = 0$

Symmetry of the pulsation equations

Consequence 3 – variational principle

- let us define a variational frequency as follows:

$$\omega_{\mathrm{var}}^2 = \frac{\left\langle \vec{\xi}, \mathcal{F}(\vec{\xi}) \right\rangle}{\left\langle \vec{\xi}, \vec{\xi} \right\rangle}$$

- according to the variational principle,

  > A small error on an eigenfunction, $\delta\vec{\xi}$, leads to a 2^{nd} (or higher) order modification of ω_{var}

- in practise, the variational principle is used to check the accuracy of the frequencies through a comparison of ω and ω_{var} (see, e.g., ADIPLS code)

Perturbing the the pulsation equations

- we now return to our perturbed pulsation equations:

$$(\delta\omega^2)\vec{\xi} + \omega^2(\delta\vec{\xi}) = \delta\mathcal{F}(\vec{\xi}) + \mathcal{F}(\delta\vec{\xi})$$

- taking the dot product between this equation and $\vec{\xi}$ yields:

$$\delta\omega^2 \left\langle \vec{\xi}, \vec{\xi} \right\rangle + \omega^2 \left\langle \vec{\xi}, \delta\vec{\xi} \right\rangle = \left\langle \vec{\xi}, \delta\mathcal{F}(\vec{\xi}) \right\rangle + \left\langle \vec{\xi}, \mathcal{F}(\delta\vec{\xi}) \right\rangle$$

- grouping terms with $\delta\vec{\xi}$ yields:

$$\begin{aligned}
\delta\omega^2 \left\langle \vec{\xi}, \vec{\xi} \right\rangle - \left\langle \vec{\xi}, \delta\mathcal{F}(\vec{\xi}) \right\rangle &= -\omega^2 \left\langle \vec{\xi}, \delta\vec{\xi} \right\rangle + \left\langle \vec{\xi}, \mathcal{F}(\delta\vec{\xi}) \right\rangle \\
&= -\omega^2 \left\langle \vec{\xi}, \delta\vec{\xi} \right\rangle + \left\langle \mathcal{F}(\vec{\xi}), \delta\vec{\xi} \right\rangle \\
&= \left\langle -\omega^2\vec{\xi}, \delta\vec{\xi} \right\rangle + \left\langle \mathcal{F}(\vec{\xi}), \delta\vec{\xi} \right\rangle \\
&= \left\langle -\omega^2\vec{\xi} + \mathcal{F}(\vec{\xi}), \delta\vec{\xi} \right\rangle \\
&= 0
\end{aligned}$$

- the right-hand side vanishes because $\vec{\xi}$ is an eigenmode, and ω^2 its associated eigenvalue.

Perturbing the the pulsation equations

- isolating $\delta\omega^2$ then yields

$$\delta(\omega^2) = 2\omega\delta\omega = \frac{\left\langle \vec{\xi}, \delta\mathcal{F}(\vec{\xi}) \right\rangle}{\left\langle \vec{\xi}, \vec{\xi} \right\rangle}$$

- this last form is extremely useful because it relates modifications of the pulsation frequency to changes in the stellar model, *without needing* $\delta\vec{\xi}$
  - hence, frequency modifications are directly related to perturbations of the stellar model

What types of perturbations?

- rotation
  - 1D case: $\Omega \equiv \Omega(r)$
  - 2D case: $\Omega \equiv \Omega(r, \theta)$
- structural $(\rho_0 + \delta\rho_0, c_0^2 + \delta c_0^2, \Gamma_{1,0} + \delta\Gamma_{1,0} \ldots)$
  - 1D case

Rotation

- rotation introduces the following effects:
  - centrifugal distortion – this leads to 2nd order effects which will be neglected
  - Coriolis force – first order effects on frequencies
- Euler's equation revisited:

$$\frac{\mathrm{d}^2\vec{\xi}}{\mathrm{d}t^2} + \vec{\xi}\cdot\vec{\nabla}\left(\vec{v}_0\cdot\vec{\nabla}\vec{v}_0\right) = -\frac{\vec{\nabla}p'}{\rho_0} + \frac{\rho'\vec{g}_{\mathrm{eff}}}{\rho_0} - \vec{\nabla}\Psi'$$

where $\frac{\mathrm{d}\vec{\xi}}{\mathrm{d}t} \equiv \frac{\partial\vec{\xi}}{\partial t} + \vec{v}_0\cdot\vec{\nabla}\vec{\xi}$ and $\vec{v}_0 = s\Omega\vec{e}_\phi$

- neglecting terms $\propto \Omega^2$ and doing various simplifications leads to:

$$-\omega^2\vec{\xi} = -2\omega m\Omega\vec{\xi} + 2i\omega\vec{\Omega}\times\vec{\xi} - \frac{\vec{\nabla}p'}{\rho_0} + \frac{\rho'\vec{g}_0}{\rho_0} - \vec{\nabla}\Psi'$$

where we've used $\vec{\xi} \propto \exp(-i\omega t + im\phi)$

- from this we deduce:

$$\delta\mathcal{F}(\vec{\xi}) = 2\omega m\Omega\vec{\xi} - 2i\omega\vec{\Omega}\times\vec{\xi}$$

Rotation

- the frequency shift is then:

$$
\begin{aligned}
\delta\omega &= \frac{1}{2\omega}\frac{\left\langle\vec{\xi},\delta\mathcal{F}(\vec{\xi})\right\rangle}{\left\langle\vec{\xi},\vec{\xi}\right\rangle} \\
&= \frac{1}{2\omega}\frac{\int_V \rho_0\vec{\xi}^*\cdot\left(2\omega m\Omega\vec{\xi}-2i\omega\vec{\Omega}\times\vec{\xi}\right)\mathrm{d}V}{\int_V \rho_0\|\vec{\xi}\|^2\mathrm{d}V} \\
&= m\frac{\int_V \rho_0\Omega\|\vec{\xi}\|^2\mathrm{d}V}{\int_V \rho_0\|\vec{\xi}\|^2\mathrm{d}V} - i\frac{\int_V \rho_0\vec{\Omega}\cdot\left(\vec{\xi}\times\vec{\xi}^*\right)\mathrm{d}V}{\int_V \rho_0\|\vec{\xi}\|^2\mathrm{d}V} \\
&= \underbrace{m\frac{\int_V \rho_0\Omega\|\vec{\xi}\|^2\mathrm{d}V}{\int_V \rho_0\|\vec{\xi}\|^2\mathrm{d}V}}_{\text{advection}}\underbrace{+2\frac{\int_V \rho_0\vec{\Omega}\cdot\left[\Im(\vec{\xi})\times\Re(\vec{\xi})\right]\mathrm{d}V}{\int_V \rho_0\|\vec{\xi}\|^2\mathrm{d}V}}_{\text{Coriolis}}
\end{aligned}
$$

Rotation – 1D case

- in the 1D case, this becomes:

$$\delta\omega = m \int_0^R K_\Omega^{n\ell}(r)\Omega(r)\mathrm{d}R$$

where:

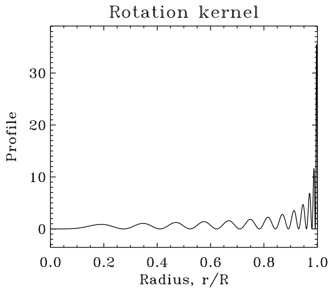
$$K_\Omega^{n\ell} = \frac{\rho_0 r^2 \left(\xi^2 + \ell(\ell+1)\eta^2 - 2\xi\eta - \eta^2\right)}{\int_{r=0}^R \rho_0(r)\left(\xi^2 + \ell(\ell+1)\eta^2\right) r^2 \mathrm{d}r}$$

$$\vec{\xi} = \xi Y_m^\ell \vec{e}_r + \eta\left(\frac{\partial Y_m^\ell}{\partial\theta}\vec{e}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_m^\ell}{\partial\varphi}\vec{e}_\phi\right)$$

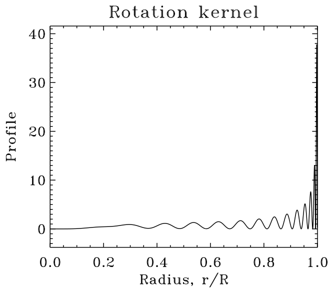
$$\ell = \text{harmonic degree}$$

- $K_\Omega^{n\ell}$ is known as the *rotation kernel*
- this expression yields a uniform splitting

Examples of 1D rotation kernels

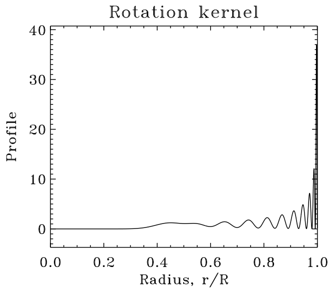


$(n, \ell) = (13, 1)$

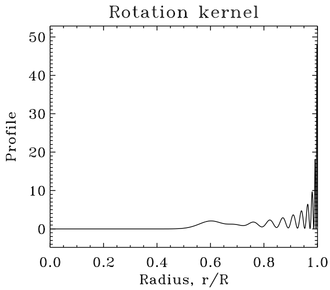


$(n, \ell) = (13, 3)$

Examples of 1D rotation kernels



$(n, \ell) = (10, 10)$



$(n, \ell) = (10, 20)$

Link with Ledoux constant

- if Ω is constant, the previous expression simplifies a bit further:

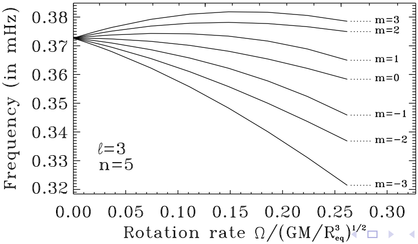
$$\delta\omega = m(1-\mathcal{C})\Omega$$

where

$$\mathcal{C} = \frac{\int_{r=0}^{R} \rho_0 \left(2\xi\eta + \eta^2\right) r^2 \mathrm{d}r}{\int_{r=0}^{R} \rho_0(r) \left(\xi^2 + \ell(\ell+1)\eta^2\right) r^2 \mathrm{d}r}$$

- $\mathcal{C}$ is the Ledoux constant (see Ledoux 1951)
- hence, to first order the frequencies of a uniformly rotating star are:

$$\omega_{n\ell m} \simeq \omega_{n\ell}^0 + m(1-\mathcal{C})\Omega$$



Rotation – 2D case

- in the 2D case, this becomes:

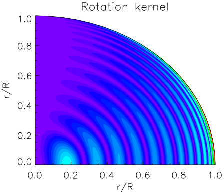
$$\delta\omega_{n,\,\ell,\,m} = \omega_{n,\,\ell,\,m} - \omega_{n,\,\ell,\,0} = \int_0^R \int_0^\pi \mathcal{K}_{n,\,\ell,\,m} \Omega(r,\theta) r \mathrm{d}r \mathrm{d}\theta$$

where:

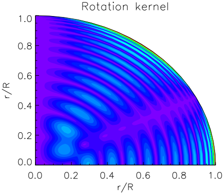
$$\mathcal{K}_{n,\,\ell,\,m} = m \frac{2\pi r \rho_0 \sin\theta \left\{ \xi^2 |Y_m^\ell|^2 + \eta^2 \left[\left| \frac{\partial Y_\ell^m}{\partial \theta} \right|^2 + \frac{m^2 |Y_m^\ell|^2}{\sin^2\theta} \right] - 2\xi\eta |Y_m^\ell|^2 - 2\eta^2 \Re \left[\frac{\cos\theta}{\sin\theta} \frac{\partial Y_\ell^m}{\partial\theta} \left(Y_m^\ell\right)^\star \right] \right\}}{\int_0^R \rho_0 \left(\xi^2 + \ell(\ell+1)\eta^2\right) r^2 dr}$$

- this time, the splitting may be non-uniform

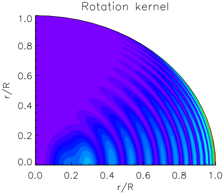
Examples of 2D rotation kernels



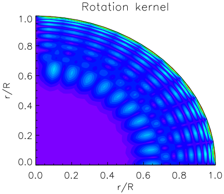
$(n, \ell, m) = (13, 1, 1)$



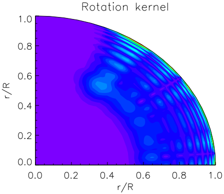
$(n, \ell, m) = (13, 3, 1)$



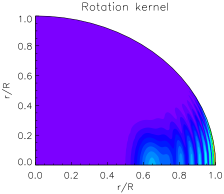
$(n, \ell, m) = (13, 3, 3)$



$(n, \ell, m) = (8, 20, 1)$



$(n, \ell, m) = (8, 20, 10)$



$(n, \ell, m) = (8, 20, 20)$

Structural kernels

- the acoustic structure of stars is typically determined by two variables, *e.g.* $(\rho_0, \Gamma_{1,0})$
- accordingly, when modifying the structure of the star, the modifications to two structural quantities need to be specified, *e.g.* $(\delta\rho_0, \delta\Gamma_{1,0})$

Structural kernels

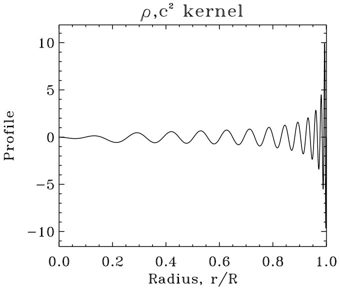
- after a (very) lengthy derivation, one can show that:

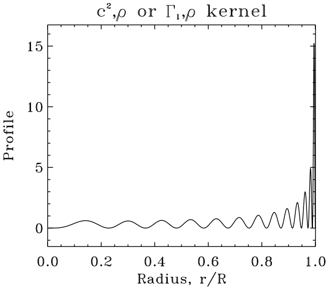
$$\frac{\delta\omega}{\omega} = \int_{r=0}^{R} \left[K_{c^2,\rho}(r) \frac{\delta c_0^2(r)}{c_0^2(r)} + K_{\rho,c^2}(r) \frac{\delta\rho_0(r)}{\rho_0(r)} \right] \mathrm{d}r$$

where:

$$\begin{aligned}
K_{c^2,\rho} &= \frac{\rho_0 c_0^2 \chi^2 r^2}{2I\omega^2} \\
K_{\rho,c^2} &= \frac{\rho_0 r^2}{2I\omega^2} \left\{ c_0^2\chi^2 - \omega^2 \left(\xi^2 + \ell(\ell+1)\eta^2\right) - 2g_0\xi\chi - 4\pi G \int_{s=r}^{R} \left(2\rho_0\xi\chi + \frac{\mathrm{d}\rho_0}{\mathrm{d}s}\xi^2\right) ds \right. \\
&\quad \left. + 2g_0\xi\frac{\mathrm{d}\xi}{\mathrm{d}r} + 4\pi G\rho_0\xi^2 + 2\left(\xi\frac{\mathrm{d}\psi}{\mathrm{d}r} + \frac{\ell(\ell+1)\eta\psi}{r}\right) \right\} \\
I &= \int_0^R \rho_0 \left(\xi^2 + \ell(\ell+1)\eta^2\right) r^2 \mathrm{d}r \\
\chi &= \frac{\vec{\nabla}\cdot\vec{\xi}}{Y_m^\ell} = \frac{\mathrm{d}\xi}{\mathrm{d}r} + \frac{2\xi}{r} - \frac{\ell(\ell+1)\eta}{r} \\
\rho &= -\frac{\mathrm{d}\rho_0}{\mathrm{d}r}\xi - \rho_0\chi \\
\psi &= -\frac{4\pi G}{2\ell+1} \left[\int_{s=0}^{r} \rho(s) \frac{s^{\ell+2}}{r^{\ell+1}} ds + \int_{s=r}^{R} \rho(s) \frac{r^\ell}{s^{\ell-1}} ds \right] \\
\frac{\mathrm{d}\psi}{\mathrm{d}r} &= -\frac{4\pi G}{2\ell+1} \left[-(\ell+1) \int_{s=0}^{r} \rho(s) \frac{s^{\ell+2}}{r^{\ell+2}} ds + \ell \int_{s=r}^{R} \rho(s) \frac{r^{\ell-1}}{s^{\ell-1}} ds \right]
\end{aligned}$$

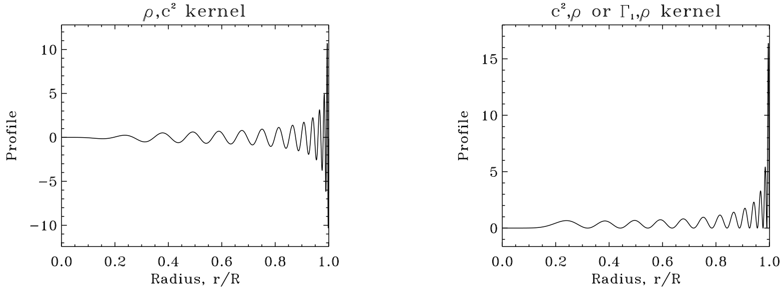
Examples of (ρ_0, c_0^2) kernels





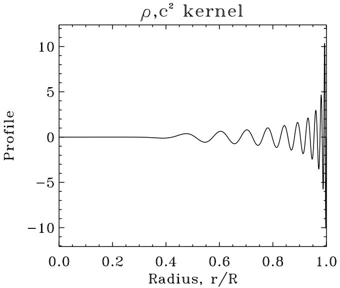
$$(n, \ell) = (13, 1)$$

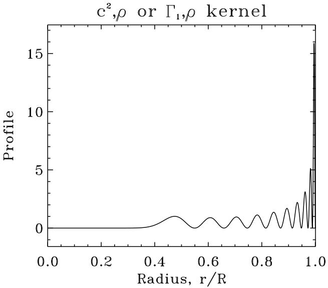
Examples of (ρ_0, c_0^2) kernels



$$(n, \ell) = (13, 3)$$

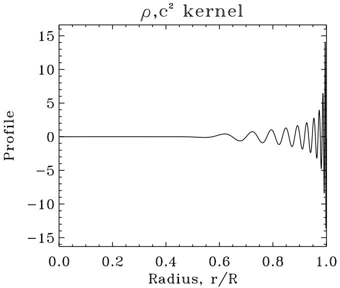
Examples of (ρ_0, c_0^2) kernels

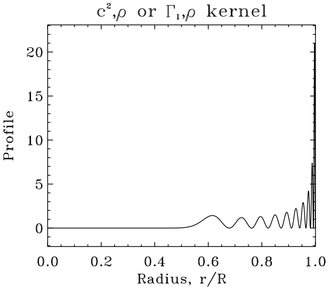




$$(n, \ell) = (10, 10)$$

Examples of (ρ_0, c_0^2) kernels





$$(n, \ell) = (10, 20)$$

Structural kernels

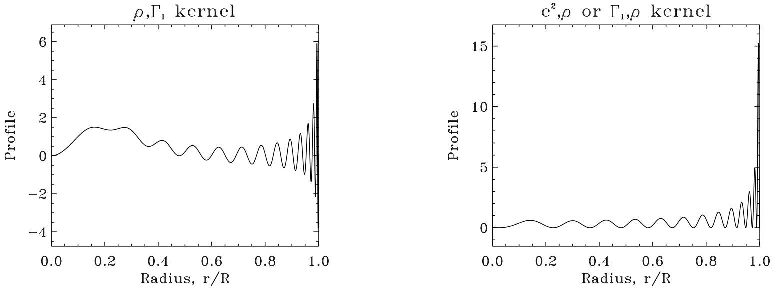
- It is possible to obtain kernels for other structural pairs
- For instance $(\rho_0, \Gamma_{1,0})$ kernels can be deduced from the (ρ_0, c_0^2) kernels, thanks to the following relation:

$$
\begin{aligned}
\frac{\delta\omega}{\omega} &= \int_{r=0}^{R} \left[K_{c^2,\rho}(r) \frac{\delta c_0^2(r)}{c_0^2(r)} + K_{\rho,c^2}(r) \frac{\delta \rho_0(r)}{\rho_0(r)} \right] \mathrm{d}r \\
&= \int_{0}^{R} \left[K_{\Gamma_1,\rho}(r) \frac{\delta \Gamma_{1,0}(r)}{\Gamma_{1,0}(r)} + K_{\rho,\Gamma_1}(r) \frac{\delta \rho_0(r)}{\rho_0(r)} \right] \mathrm{d}r
\end{aligned}
$$

  along with the perturbed expression for hydrostatic equilibrium
- after various permutations of integrals, the following expressions are obtained:

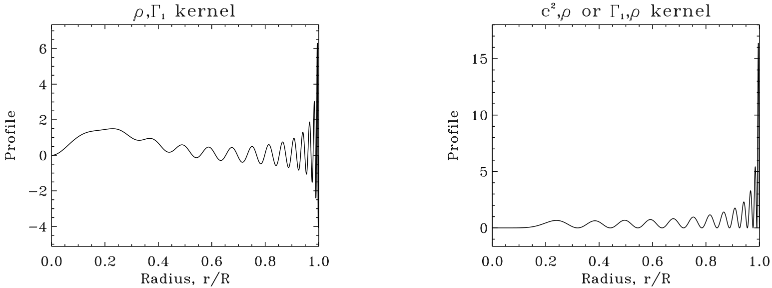
$$
\begin{aligned}
K_{\Gamma_1,\rho} &= K_{c^2,\rho} = \frac{\rho_0 c_0^2 \chi^2 r^2}{2I\omega^2} \\
K_{\rho,\Gamma_1} &= K_{\rho,c^2} - K_{c^2,\rho} + \frac{Gm\rho_0}{r^2} \int_{s=0}^{r} \frac{K_{c^2,\rho}(s)}{p_0(s)} ds + \rho_0 r^2 \int_{s=r}^{R} \frac{4\pi G\rho_0}{s^2} \left(\int_{t=0}^{s} \frac{K_{c^2,\rho}(t)}{p_0(t)} dt \right) ds \\
&= K_{\rho,c^2} - K_{c^2,\rho} + \frac{Gm\rho_0}{r^2} \int_{s=0}^{r} \frac{\Gamma_{1,0}\chi^2 s^2}{2I\omega^2} ds + \rho_0 r^2 \int_{s=r}^{R} \frac{4\pi G\rho_0}{s^2} \left(\int_{t=0}^{s} \frac{\Gamma_{1,0}\chi^2 t^2}{2I\omega^2} dt \right) ds
\end{aligned}
$$

Examples of $(\rho_0, \Gamma_{1,0})$ kernels



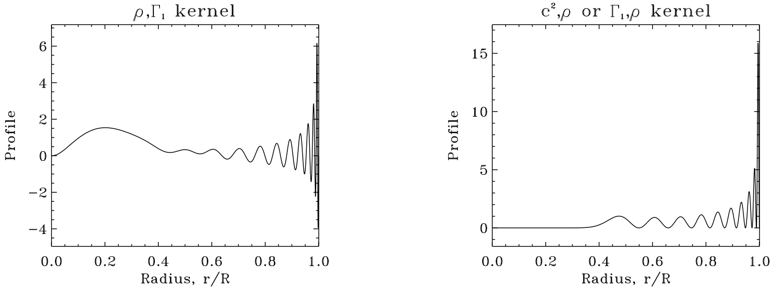
$$(n, \ell) = (13, 1)$$

Examples of $(\rho_0, \Gamma_{1,0})$ kernels



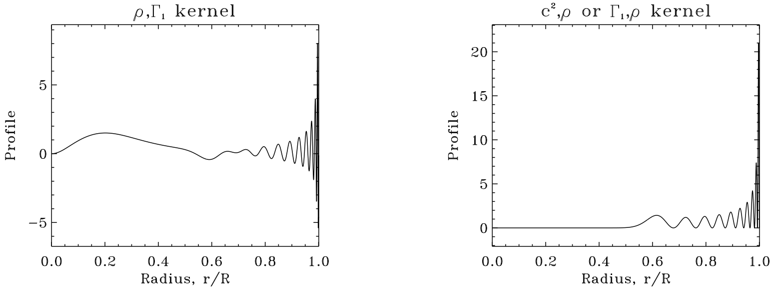
$$(n, \ell) = (13, 3)$$

Examples of $(\rho_0, \Gamma_{1,0})$ kernels



$$(n, \ell) = (10, 10)$$

Examples of $(\rho_0, \Gamma_{1,0})$ kernels

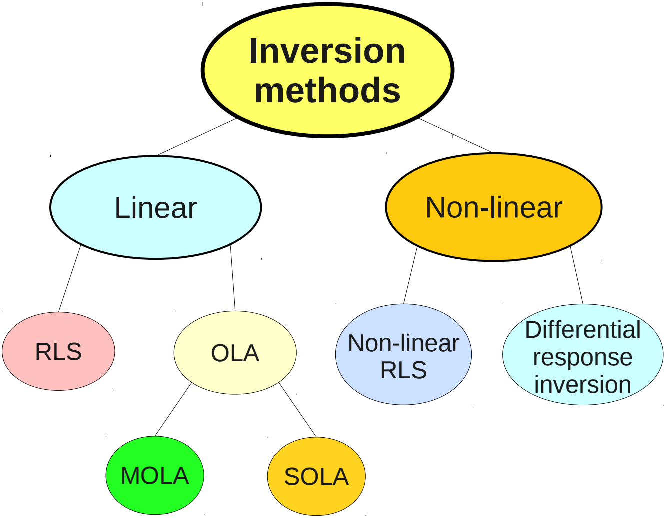


$$(n, \ell) = (10, 20)$$

Other structural kernels

- other structural kernels an be obtained:
  - $(u \equiv \frac{P}{\rho}, \Gamma_1)$, (g, Γ_1), (P, Γ_1), (u, Y), (A, Γ_1), (N^2, c^2) etc. (see Masters 1979, Gough & Thompson, 1991, Elliott, 1996, Basu & Christensen-Dalsgaard, 1997, Kosovichev, 1999, Buldgen et al., in prep)
  - some of these require the equation of state and its derivatives

Classification of inversion techniques



What can be inverted

- rotation profile
- structural profiles
- integrated quantities

Rotation inversions

The linearised inversion problem

$$
\begin{aligned}
S_{n_1,\ell_1} &= \frac{\nu_{n_1,\ell_1,m_1} - \nu_{n_1,\ell_1,0}}{m_1} = \int_0^R K_\Omega^{n_1,\ell_1}(r)\Omega(r)\mathrm{d}r + \varepsilon_{n_1,\ell_1} \\
S_{n_2,\ell_2} &= \frac{\nu_{n_2,\ell_2,m_2} - \nu_{n_2,\ell_2,0}}{m_2} = \int_0^R K_\Omega^{n_2,\ell_2}(r)\Omega(r)\mathrm{d}r + \varepsilon_{n_2,\ell_2} \\
S_{n_3,\ell_3} &= \frac{\nu_{n_3,\ell_3,m_3} - \nu_{n_3,\ell_3,0}}{m_3} = \int_0^R K_\Omega^{n_3,\ell_3}(r)\Omega(r)\mathrm{d}r + \varepsilon_{n_3,\ell_3} \\
\dots
\end{aligned}
$$

where:

$$
\begin{aligned}
S_{n,\ell} &= \text{are the “rotational splittings”} = \text{the observations} \\
\Omega(r) &= \text{the rotation profile} = \text{the unknown} \\
\varepsilon_{n,\ell} &= \text{observational error realisations } (\langle \varepsilon_{n,\ell} \rangle = \sigma_{n,\ell})
\end{aligned}
$$

Note: in what follows, we will use the index "l" as shorthand for (n, ℓ).

Rotation inversions

Goal

$$\underbrace{S_l}_{\text{obs.}} = \int_0^R \underbrace{K^l_\Omega(r)}_{\text{known}} \underbrace{\Omega(r)}_{\text{unknown}} \, \mathrm{d}r + \varepsilon_l$$

- Find $\Omega(r)$ from the S_l, *i.e.* invert above integral relations

Rotation inversions

Goal

$$\underbrace{S_l}_{\text{obs.}} = \int_0^R \underbrace{K_\Omega^l(r)}_{\text{known}} \underbrace{\Omega(r)}_{\text{unknown}} \, \mathrm{d}r + \varepsilon_l$$

- Find $\Omega(r)$ from the S_l, *i.e.* invert above integral relations

- at first look, this problem looks impossible:
  - the unknown is a function
  - only a finite number of observations/constraints
  - problem is ill-posed (as we will see later on)

- the solution to the above problems involves injecting *a priori* assumptions
  - accordingly, we should always bear in mind these limitations when looking at inversion results

Rotation inversions

A first approach

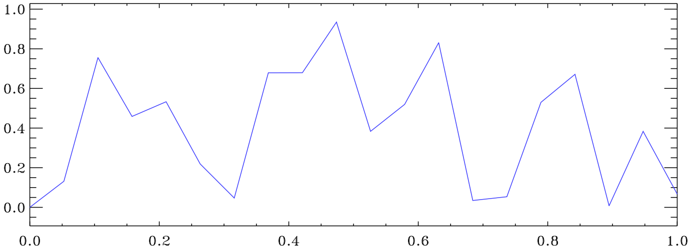
- write solution in terms of basis functions:

$$\Omega_{\mathrm{inv}}(r) = \sum_k a_k \phi_k(r)$$

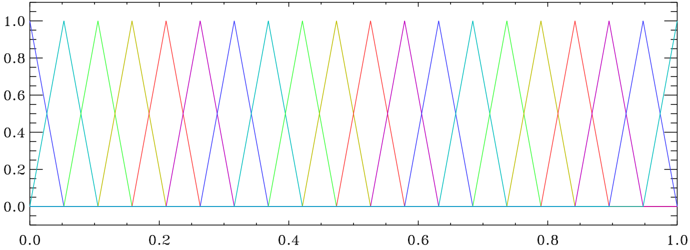
  where a_k are unknown coefficients, and ϕ_k basis functions
- typical choices for ϕ_k include b-splines functions of various degrees
  - **degree=0:** step-wise function
  - **degree=1:** "connect-the-dots" function
  - **degree=3:** cubic b-spline function

"Connect-the-dots" function (degree=1)

Function

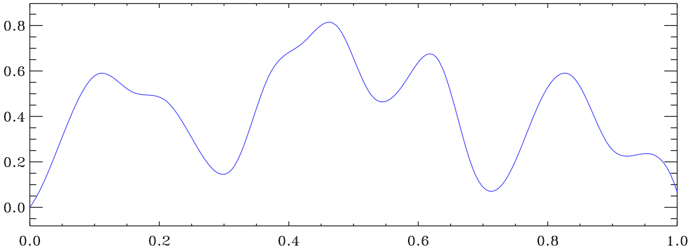


Basis

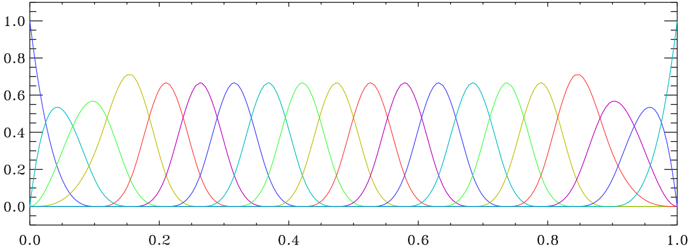


Cubic b-spline function (degree=3)

Function



Basis



Least-squares approach

A first approach

- substitute above expression into kernel expression:

$$\tilde{S}_l = \int_0^R K_\Omega^l(r)\Omega_{\mathrm{inv}}(r)\mathrm{d}r$$

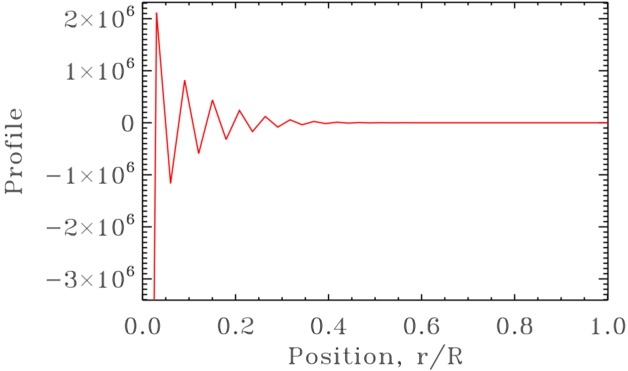
- minimise distance between S_l and $\tilde{S}_l$:

$$\min J(a_k) = \sum_l \frac{\left(S_l - \tilde{S}_l\right)^2}{\sigma_l^2}$$

- a least squares solution is obtained by solving $\vec{\nabla} J = \vec{0}$:

$$\sum_l \frac{S_l}{\sigma_l^2} \int_0^R K_\Omega^l(r)\phi_k(r)\mathrm{d}r = \sum_{k'} \left[\sum_l \frac{1}{\sigma_l^2} \int_0^R K_\Omega^l(r)\phi_k(r)\mathrm{d}r \int_0^R K_\Omega^l(r)\phi_{k'}(r)\mathrm{d}r \right] a_{k'}$$

Least-squares approach – an inversion result



- there are 830 rotational splittings
- $\Omega_{\rm inv}$ is described using 50 b-splines

Regularised Least-Squares approach (RLS)

- clearly the above approach produced a very poor solution
  - such inversion problems are usually ill-conditioned (= high sensitivity to noise)
- one remedy is to "regularise" the solution. This can be done by introducing a supplementary term to the cost function:

$$J(a_k) = \sum_l \frac{\left(S_l - \tilde{S}_l\right)^2}{\sigma_l^2} + \Lambda \left\langle \frac{1}{\sigma^2} \right\rangle \int_0^R \left(\frac{\mathrm{d}^2\Omega_{\mathrm{inv}}}{\mathrm{d}r^2} \right)^2 \mathrm{d}r$$

  where $\left\langle \frac{1}{\sigma^2} \right\rangle = \frac{1}{L} \sum_l \frac{1}{\sigma_l^2}$

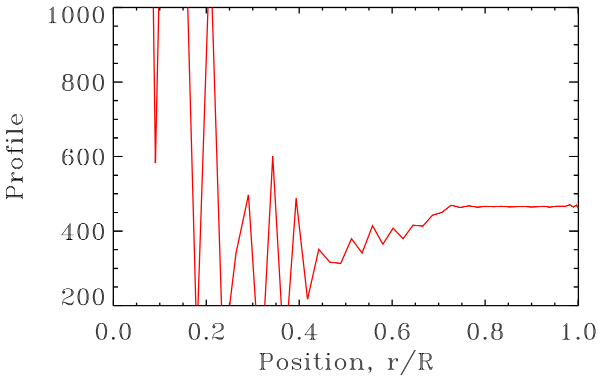
- Λ is a regularisation parameter:
  - small values of Λ = less regularisation, but closer fit to S_l
  - large values of Λ = more regularisation, but worse fit to S_l

Regularised Least-Squares approach (RLS)

- solving $\vec{\nabla} J = \vec{0}$ leads to the following set of equations:

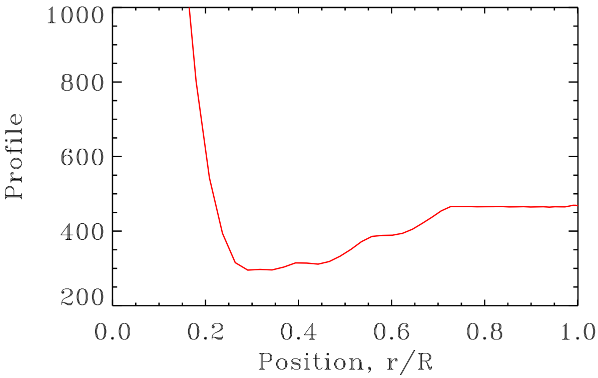
$$\sum_l \frac{S_l}{\sigma_l^2} \int_0^R K_\Omega^l(r)\phi_k(r)\mathrm{d}r = \sum_{k'} \left[\sum_l \frac{1}{\sigma_l^2} \int_0^R K_\Omega^l(r)\phi_k(r)\mathrm{d}r \int_0^R K_\Omega^l(r)\phi_{k'}(r)\mathrm{d}r + \Lambda \left\langle \frac{1}{\sigma^2} \right\rangle \int_0^R \frac{\mathrm{d}^2\phi_k}{\mathrm{d}r^2} \frac{\mathrm{d}^2\phi_{k'}}{\mathrm{d}r^2} \mathrm{d}r \right] a_{k'}$$

Regularised Least-Squares approach (RLS) – an inversion result



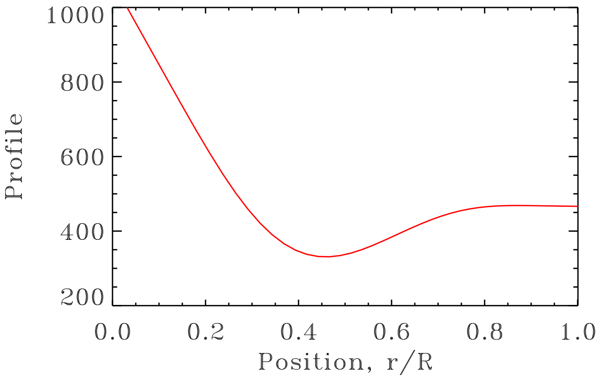
$$\Lambda = 10^{-13}$$

Regularised Least-Squares approach (RLS) – an inversion result



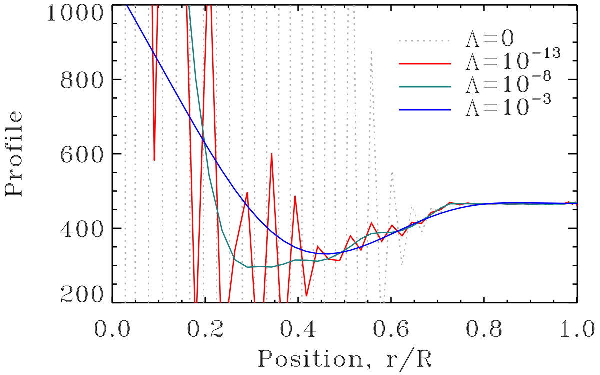
$$\Lambda = 10^{-8}$$

Regularised Least-Squares approach (RLS) – an inversion result



$\Lambda = 10^{-3}$

Regularised Least-Squares approach (RLS) – an inversion result



$$\Lambda_{\mathrm{opt.}} = 10^{-8}$$

Error propagation

- let r_0 be a given grid point
- the relationship between $\Omega_{\mathrm{inv}}(r_0)$ and a_k is linear:

$$\Omega_{\mathrm{inv}}(r_0) = \sum_k a_k \phi_k(r_0)$$

- furthermore, the relationship between the a_k and the S_l is linear
- hence, the relationship between $\Omega_{\mathrm{inv}}(r_0)$ and the S_l is linear. This relationship is expressed as follows:

$$\Omega_{\mathrm{inv}}(r_0) = \sum_l c_l(r_0) S_l$$

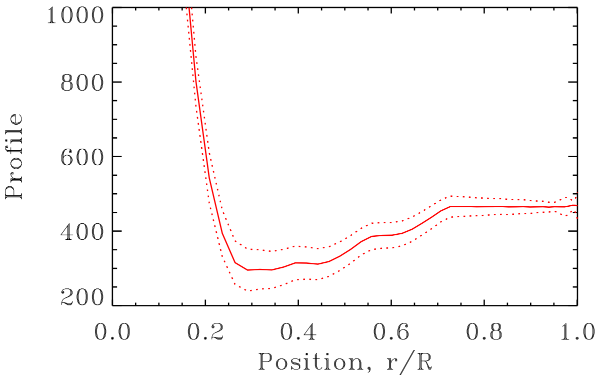
Error propagation

- assuming the errors, ε_l, are uncorrelated, then the error bar on $\Omega_{\mathrm{inv}}(r_0)$ is:

$$\sigma_{\Omega(r_0)} = \sqrt{\sum_l (c_l \sigma_l)^2}$$

- This only takes into account how the observational errors propagate through the inversion. It doesn't take into account
  - poorly adjusted averaging kernels
  - departures from linearity

Error propagation



$$\Lambda = 10^{-8}$$

Error magnification

- if the the error bars are uniform, $\sigma_l \equiv \sigma$, the propagated error becomes:

$$\sigma_{\Omega(r_0)} = \sqrt{\sum_l (c_l \sigma)^2} = \sigma \sqrt{\sum_l (c_l)^2}$$

- the quantity $\sqrt{\sum_l (c_l)^2}$ is known as the *error magnification*

Averaging kernel

- we recall the linear relation between S_l and $\Omega_{\mathrm{inv}}(r_0)$:

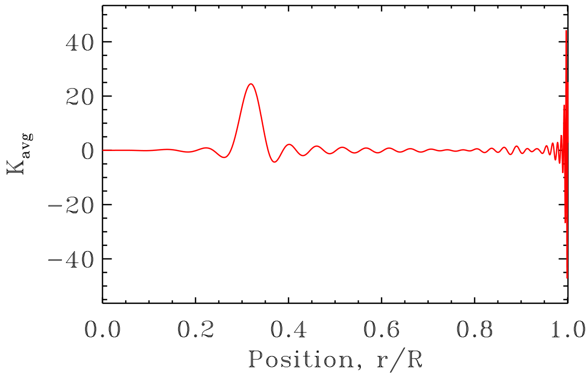
$$\Omega_{\mathrm{inv}}(r_0) = \sum_l c_l(r_0) S_l$$

- we replace S_l by its kernel-based expression:

$$\begin{aligned}
\Omega_{\mathrm{inv}}(r_0) &= \sum_l c_l(r_0) \left[\int_0^R K_\Omega^l(r)\Omega(r)\mathrm{d}r + \varepsilon_l\right] \\
&= \int_0^R \underbrace{\sum_l c_l(r_0) K_\Omega^l(r)}_{\mathcal{K}_{\mathrm{avg}}(r_0,r)} \Omega(r)\mathrm{d}r + \sum_l c_l(r_0)\varepsilon_l
\end{aligned}$$

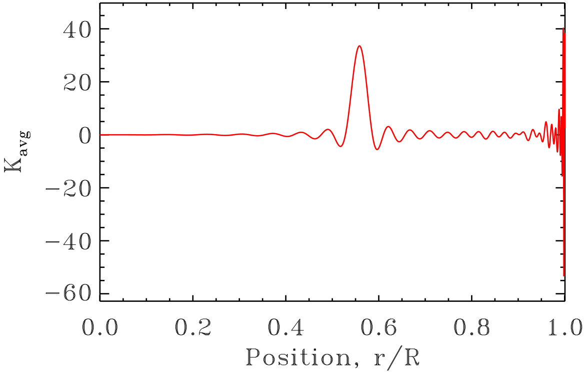
- This expression shows that $\Omega_{\mathrm{inv}}(r_0)$ is in fact an average of the true rotation profile $\Omega(r)$. The corresponding weight function, $\mathcal{K}_{\mathrm{avg}}(r_0, r)$, is the known as the *averaging kernel*.

Averaging kernels



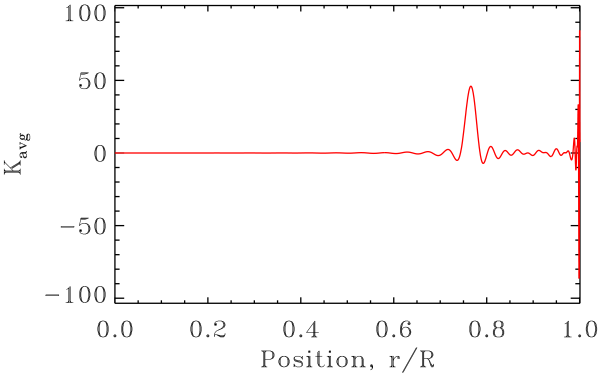
$$r_0 = 0.3171R$$

Averaging kernels



$$r_0 = 0.5578R$$

Averaging kernels



$r_0 = 0.7652R$

Optimally Localised Averages (OLA)

- **Goal**: optimise the averaging kernels
- two variants:
  - **MOLA**: Multiplicative OLA
  - **SOLA**: Subtractive OLA

MOLA – Multiplicative OLA

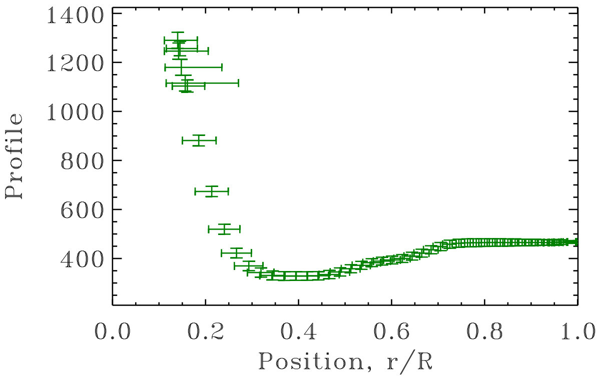
- reference: Backus & Gilbert (1968)

$$J(c_l(r_0)) = \underbrace{\int_0^R P(r_0,r)\,[\mathcal{K}_{\mathrm{avg}}(r_0,r)]^2 \mathrm{d}r}_{\text{fit data}} + \underbrace{\frac{\tan\theta}{\langle\sigma^2\rangle}\sum_{l=1}^{L}(c_l\sigma_l)^2}_{\text{regularisation}} + \underbrace{\lambda\left\{1-\int_0^R \mathcal{K}_{\mathrm{avg}}\right\}}_{\mathcal{K}_{\mathrm{avg}}\ \text{unimodular}}$$

where:

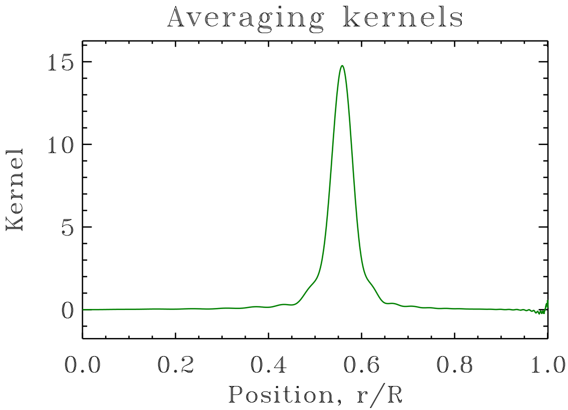
$$\begin{aligned}
\left\langle\sigma^2\right\rangle &= \frac{1}{L}\sum_{l=1}^{L}\sigma_l^2 \\
\theta &= \text{trade-off parameter between fitting data and reducing error,} \\
P(r_0,r) &= \text{penalty function (usually } 12(r-r_0)^2\text{)} \\
\lambda &= \text{Lagrangian multiplier used to ensure } \int_0^R \mathcal{K}_{\mathrm{avg}}(r_0,r)\mathrm{d}r = 1
\end{aligned}$$

MOLA inversion result



$$\theta = 10^{-2}$$

MOLA averaging kernel



$r_0 = 0.5578$

SOLA – Subtractive OLA

- references: Pijpers & Thompson (1992, 1994)

$$J(c_l(r_0)) = \int_0^R \Big[\underbrace{\mathcal{T}(r_0,r)}_{\text{target}} - \mathcal{K}_{\mathrm{avg}}(r_0,r)\Big]^2 \mathrm{d}r + \underbrace{\frac{\tan\theta}{\langle\sigma^2\rangle}\sum_{l=1}^{L}(c_l\sigma_l)^2}_{\text{regularisation}} + \lambda \underbrace{\left\{1 - \int_0^R \mathcal{K}_{\mathrm{avg}}\right\}}_{\mathcal{K}_{\mathrm{avg}}\ \text{unimodular}}$$

where:

$$\begin{aligned} \theta &= \text{trade-off parameter between fitting data and reducing error,} \\ &\quad\ \text{and which can be adjusted by the user} \\ T(r_0,r) &= \text{a target function} \\ \lambda &= \text{Lagrangian multiplier used to ensure } \int_0^R \mathcal{K}_{\mathrm{avg}}(r_0,r)\mathrm{d}r = 1 \end{aligned}$$

SOLA – the target function

- ideally, the target function should be a Dirac function
- however, this is impossible with the limited number of kernels, and would lead to poor numerical results
- a Gaussian function is therefore typically chosen:

$$\mathcal{T}(r_0, r) = \frac{1}{A} \exp\left(-\frac{(r-r_0)^2}{2\Delta(r_0)^2}\right)$$

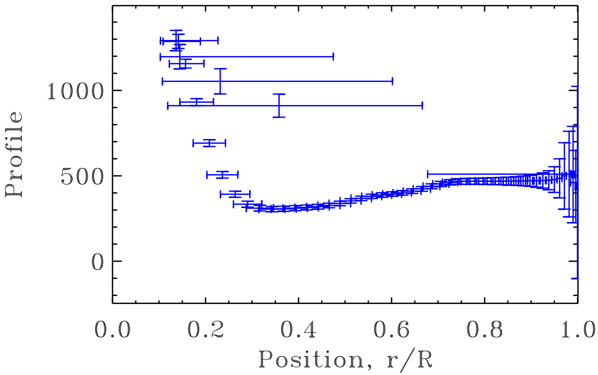
where

$$\begin{aligned} A &= \text{normalisation constant} \\ \Delta(r_0) &= \text{a function which gives the target width} \end{aligned}$$

- a good choice for $\Delta(r_0)$ when dealing with acoustic modes is (e.g. Thompson 1993):

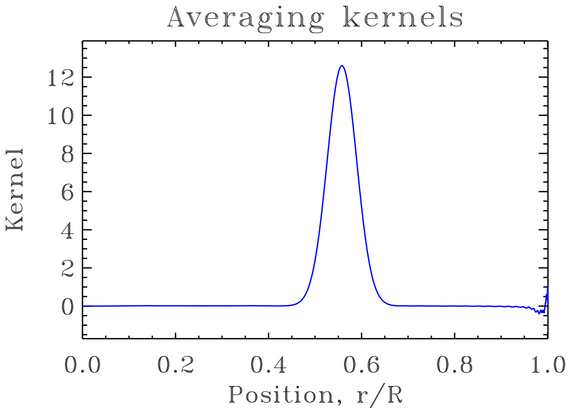
$$\Delta(r_0) \propto c_0(r_0)$$

SOLA inversion result



$$\theta = 10^{-2},\ \delta = 7 \times 10^{-2}$$

SOLA averaging kernel



$r_0 = 0.5578$

Comparison between MOLA and SOLA

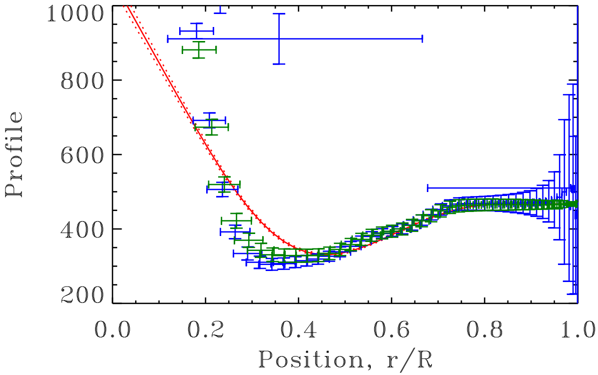
MOLA – advantages

- no width parameter to adjust
- can yield better results

SOLA – advantages

- less computationally expensive (1 matrix inversion for complete inversion)
- lends itself to inverting integrated quantities (see following slides)

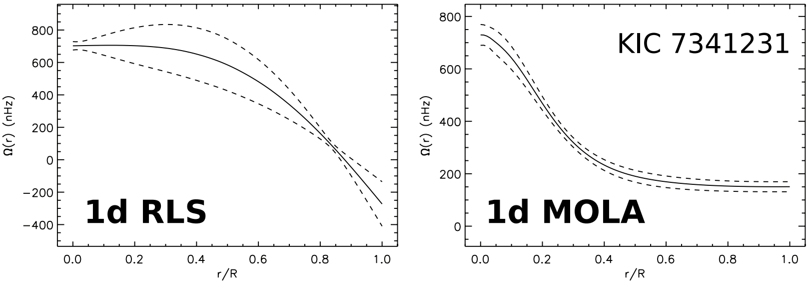
Comparison of different inversion methods



RLS SOLA MOLA

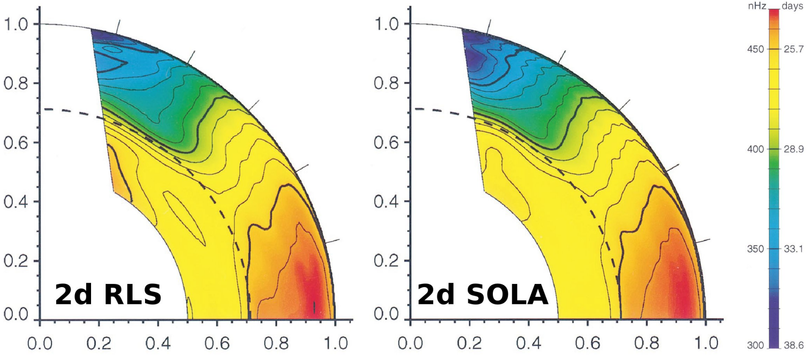
- it is important to compare different inversion methods to extract robust features

Rotation inversions in a sub-giant



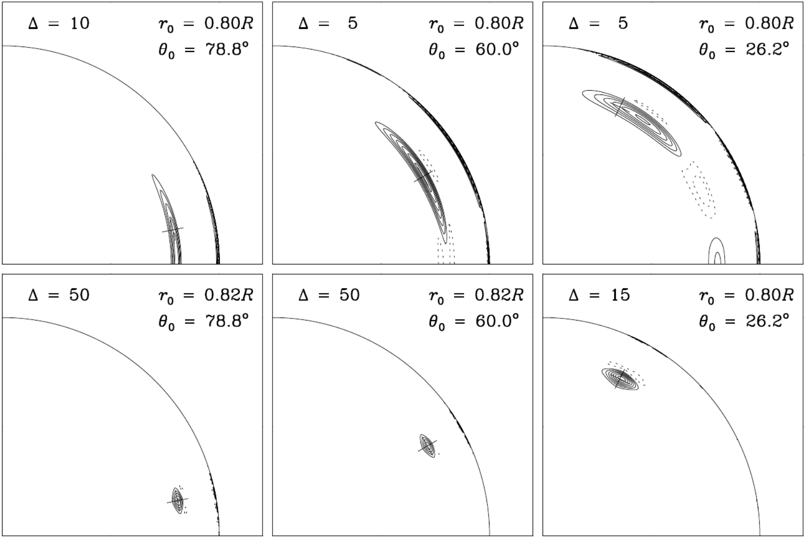
Rotation profile of a sub-giant
(Deheuvels et al., 2012)

2D rotation inversions in the sun



The solar rotation profile
(Schou et al., 1998, see also Thompson et al., 2003)

Examples of 2D averaging kernels



(adapted from Schou et al., 1994)

Structural inversions

The linearised inversion problem

$$\underbrace{\frac{\delta\nu_l}{\nu_l}}_{\text{obs.}} = \int_0^R \underbrace{K^l_{a,b}(r)}_{\text{known}} \underbrace{\frac{\delta a}{a}}_{\text{unknown}} \mathrm{d}r + \int_0^R \underbrace{K^l_{b,a}(r)}_{\text{known}} \underbrace{\frac{\delta b}{b}}_{\text{unknown}} \mathrm{d}r + \frac{F_{\mathrm{surf.}}(\nu_l)}{E_l}$$

where (a, b) are two structural profiles (e.g. (ρ, Γ_1))

- this time there are **two** functions to invert
- this leads to various modifications in the RLS and SOLA methods, as well as the introduction of $\mathcal{K}_{\mathrm{cross}}$

Structural inversions – RLS

- instead of solving this:

$$J(\Omega_{\mathrm{inv}}) = \sum_l \frac{1}{\sigma_l^2}\left(S_l - \int_0^R K_\Omega^l \Omega \mathrm{d}r\right)^2 + \Lambda \left\langle \frac{1}{\sigma^2}\right\rangle \int_0^R \left(\frac{\mathrm{d}^2\Omega}{\mathrm{d}r^2}\right)^2 \mathrm{d}r$$

- one has to solve this:

$$\begin{aligned} J\left(\frac{\delta a}{a}, \frac{\delta b}{b}\right) \quad = \quad & \sum_l \frac{1}{\sigma_l^2}\left(\frac{\delta \nu_l}{\nu_l} - \int_0^R K_{a,b}^l \frac{\delta a}{a}\mathrm{d}r - \int_0^R K_{b,a}^l \frac{\delta b}{b}\mathrm{d}r\right)^2 \\ & + \Lambda \left\langle \frac{1}{\sigma^2}\right\rangle \int_0^R \left[\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}\frac{\delta a}{a}\right)^2 + \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}\frac{\delta b}{b}\right)^2\right] \mathrm{d}r \end{aligned}$$

- one can optionally include additional terms to reduce surface effects

Averaging and cross-term kernels

- in much the same way as before, $(\delta a(r_0)/a(r_0))_{\mathrm{inv}}$ and $(\delta b(r_0)/b(r_0))_{\mathrm{inv}}$ are related in a linear way to the observables$(\delta\nu/\nu)_l$. This leads to:

$$\left(\frac{\delta a}{a}\right)_{\mathrm{inv}} = \sum_l c_l(r_0)\left(\frac{\delta\nu}{\nu}\right)_l \qquad \left(\frac{\delta b}{b}\right)_{\mathrm{inv}} = \sum_l c'_l(r_0)\left(\frac{\delta\nu}{\nu}\right)_l$$

- this leads to the following definitions:

$$\mathcal{K}_{\mathrm{avg}}(r_0, r) = \sum_{l=1}^{L} c_l(r_0) K^l_{a,b}(r) \qquad \mathcal{K}_{\mathrm{cross}}(r_0, r) = \sum_{l=1}^{L} c_l(r_0) K^l_{b,a}(r)$$

$$\mathcal{K}'_{\mathrm{avg}}(r_0, r) = \sum_{l=1}^{L} c'_l(r_0) K^l_{b,a}(r) \qquad \mathcal{K}'_{\mathrm{cross}}(r_0, r) = \sum_{l=1}^{L} c'_l(r_0) K^l_{a,b}(r)$$

Structural inversions – SOLA

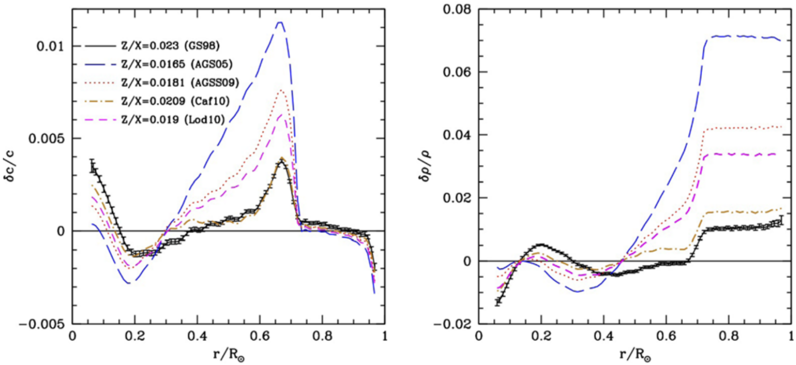
- instead of solving this:

$$J(c_l(r_0)) = \int_0^R \left[\mathcal{T}(r_0,r) - \mathcal{K}_{\text{avg}}(r_0,r)\right]^2 \mathrm{d}r + \frac{\tan\theta}{\langle\sigma^2\rangle}\sum_{l=1}^{L}(c_l\sigma_l)^2 + \lambda\left\{1 - \int_0^R \mathcal{K}_{\text{avg}}\right\}$$

- one has to solve these:

$$\begin{aligned}
J(c_l(r_0)) &= \int_0^R \left\{\mathcal{T}(r_0,r) - \mathcal{K}_{\text{avg}}(r_0,r)\right\}^2 \mathrm{d}r + \beta \int_0^R \left\{\mathcal{K}_{\text{cross}}(r_0,r)\right\}^2 \mathrm{d}r \\
&+ \frac{\tan\theta \sum_{l=1}^{L}(c_l(r_0)\sigma_l)^2}{\langle\sigma^2\rangle} + \lambda\left\{1 - \int_0^R \mathcal{K}_{\text{avg}}(r_0,r)\mathrm{d}r\right\} \\
J'(c'_l(r_0)) &= \beta' \int_0^R \left\{\mathcal{K}'_{\text{cross}}(r_0,r)\right\}^2 \mathrm{d}r + \int_0^R \left\{\mathcal{T}'(r_0,r) - \mathcal{K}'_{\text{avg}}(r_0,r)\right\}^2 \mathrm{d}r \\
&+ \frac{\tan\theta' \sum_{l=1}^{L}(c'_l(r_0)\sigma_l)^2}{\langle\sigma^2\rangle} + \lambda'\left\{1 - \int_0^R \mathcal{K}'_{\text{avg}}(r_0,r)\mathrm{d}r\right\}
\end{aligned}$$

The solar abundance problem



(Basu et al. 2014)

The solar abundance problem

Mixture	Z/X	R_{CZ}	Y_{CZ}	Y_0
Helioseismic	–	0.713 ± 0.001[a]	0.2485 ± 0.0034[b]	0.273 ± 0.006[c]
GS98	0.023	0.7139	0.2456	0.2755
AGS05	0.0165	0.7259	0.2286	0.2586
AGSS09	0.018	0.7205	0.2352	0.2650
CAF10	0.0209	0.7150	0.2415	0.2711
LOD10	0.019	0.7136	0.2412	0.2665

[a] Basu and Antia (1997)

[b] Basu and Antia (2004)

[c] Serenelli and Basu (2010)

(taken from Basu et al. 2014)

Integrated quantities

- structural inversions are difficult for stars other than the Sun, due to the limited number of modes (*e.g.* Basu et al. 2002)
- one strategy is to invert stellar parameters rather than structural profiles

Integrated quantities

- structural inversions are difficult for stars other than the Sun, due to the limited number of modes (*e.g.* Basu et al. 2002)
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How does it work?

$$\frac{\delta\rho_{\text{inv}}(r_0)}{\rho(r_0)} = \int_0^R \mathcal{K}_{\text{avg}}(r_0, r)\frac{\delta\rho}{\rho}\mathrm{d}r + \int_0^R \mathcal{K}_{\text{cross}}(r_0, r)\frac{\delta\Gamma_{1,0}}{\Gamma_{1,0}}\mathrm{d}r$$

- an inversion gives you a weighted average of the underlying profile
- **idea**: directly search for the appropriate weighting which yields the stellar parameter
- carry out a SOLA inversion with a suitable target function:

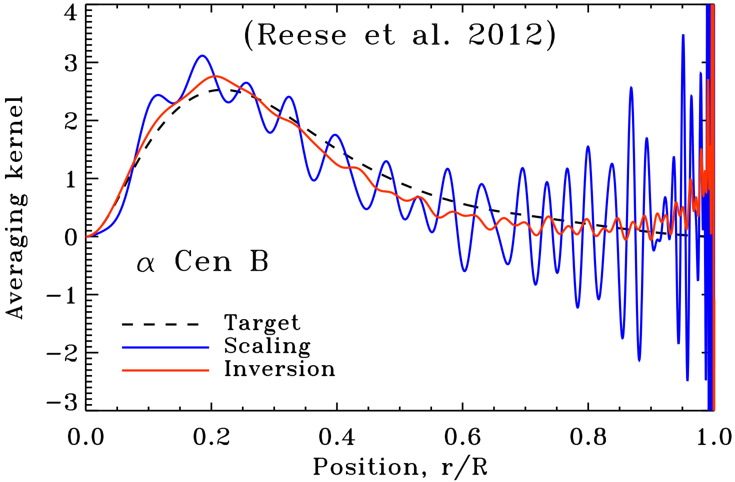
$$\text{Target function} = \frac{4\pi r^2 \rho R}{M} \qquad \Rightarrow \qquad \text{stellar mean density}$$

Integrated quantities

What parameters are accessible?

- total angular momentum (Pijpers, 1998)
- mean density (Reese et al. 2012)
- acoustic radius, core indicators (Buldgen et al. 2013)

Some examples



Non-linear inversion methods

- a second strategy for structural inversions in stars other than the Sun
- useful for stars with mixed modes which are highly sensitive to structural changes
- applies even when the reference model is far away from true structure
- still needs to be further developed

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Two approaches

- frequency-based approach
- approach based on internal phase-shifts

Non-linear RLS

Description

- iterated RLS inversions
- minimisation of the following cost function:

$$J(f) = \sum_i \left(\frac{\nu_i^{\mathrm{obs}} - \nu_i^{\mathrm{theo}}(f)}{\sigma_i} \right)^2 + \Lambda \int_0^{R_{\mathrm{cut}}} \left(\frac{\partial^2 (\ln \rho)}{\partial r^2} \right)^2 \mathrm{d}r$$

Non-linear RLS

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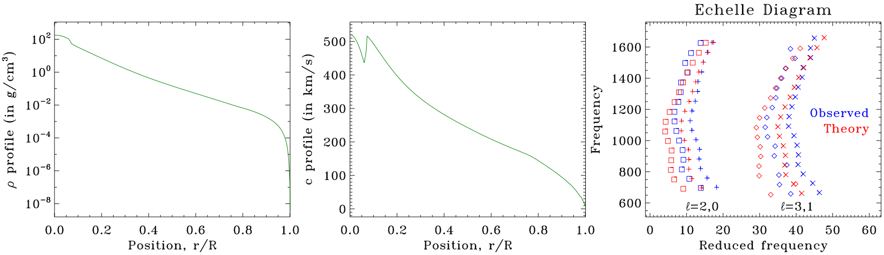
Different works

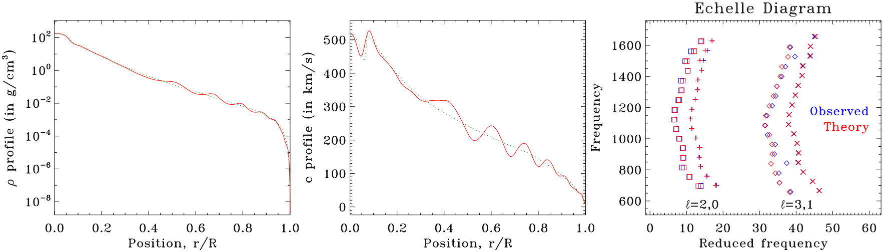
- **Antia (1996)**: inversion on $(\rho, \Gamma_{1,0})$, regularisation of $\left(\frac{\delta\rho}{\rho}, \frac{\delta\Gamma_{1,0}}{\Gamma_{1,0}}\right)$
- **Reese (ongoing)**: inversion on ρ, fixed $\Gamma_{1,0}$ profile, regularisation of ρ

Some examples

Original model



Logarithmic regularisation



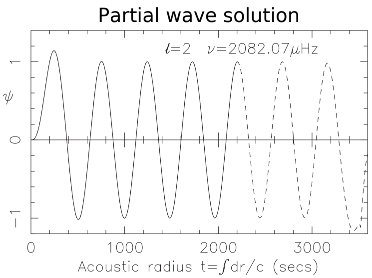
Differential Response Inversion

Description

1. Discretise $(\rho, \Gamma_{1,0})$ profiles up to a truncation point.
2. At the observed frequencies, obtain partial wave solutions and associated phase shifts.
3. Adjust model so that phase shifts become a function of frequency only.

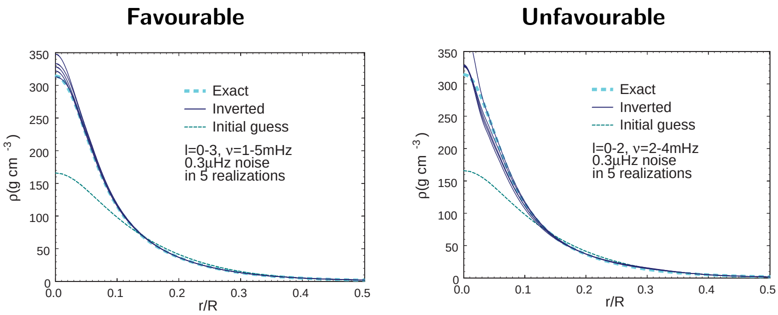
Various articles

- Vorontsov (1998, 2001), Roxburgh (2002, 2010)



(Roxburgh & Vorontsov, 2003)

An example



(Roxburgh, 2002)

- multiple realisations are used to determine the uncertainty on the profile

Conclusion

What inversions can do

- probe rotation profiles
- probe the internal structure of stars
- calculate various stellar parameters
- test new physics outside a given grid of stellar models

Limitations

- (local) linearity of the relation between pulsation frequencies and stellar structure
- there are a priori assumptions which go into constructing rotation/structural profiles
- cannot give more information than what is available in the modes

Conclusion

Recommended reading

- **Christensen-Dalsgaard's lecture notes**: the variational principle
- **Lynden-Bell & Ostriker (1967)**: the variational principle in a general context
- **Gough & Thompson (1990)**: structural kernels
- **Christensen-Dalsgaard et al. (1990)**: error propagation/magnification, averaging kernels
    - examples in this course are base on data from this article
- **Reese et al. (2012), Buldgen et al. (2015, 2016)**: inversions of stellar parameters

Inversion tools

- InversionKit: 1D inversions on individual stars
- InversionPipeline: inversions of stellar parameters using a grids of models
- NonLinearKit: experimental non-linear 1D inversion tool
- SOLA Pack (link?): 2D rotation inversions in the sun

Symmetry of the pulsation equations

Consequence 3 – variational principle

- we start with a partial solution $(\vec{\xi}, p', \Psi')$ and define the following polynomial

$$P(x) = x^2 \left\langle \vec{\xi}, \vec{\xi} \right\rangle - \left\langle \vec{\xi}, \mathcal{F}(\vec{\xi}) \right\rangle$$

- *note:* the above polynomial is defined even if $\vec{\xi}$ is not an eigenmode
- solving the equation $P(x) = 0$ leads to two solutions: $\pm X$
  - if $\vec{\xi}$ is an eigenmode, then X is the associated pulsation frequency ω

Symmetry of the pulsation equations

Consequence 3 – variational principle

- How does X vary when the $\vec{\xi}$ is slightly modified?
- We calculate the differential of $P(X) = 0$:

$$2X\delta X \left\langle \vec{\xi}, \vec{\xi} \right\rangle + X^2 \left\{ \left\langle \delta\vec{\xi}, \vec{\xi} \right\rangle + \left\langle \vec{\xi}, \delta\vec{\xi} \right\rangle \right\} = \left\{ \left\langle \delta\vec{\xi}, \mathcal{F}(\vec{\xi}) \right\rangle + \left\langle \vec{\xi}, \mathcal{F}(\delta\vec{\xi}) \right\rangle \right\}$$

- the term $\left\langle \vec{\xi}, \mathcal{F}(\delta\vec{\xi}) \right\rangle$ may be rewritten as $\left\langle \mathcal{F}(\vec{\xi}), \delta\vec{\xi} \right\rangle$
- the terms in curly brackets are simplified using the fact that $z + z^* = 2\Re(z)$, for any complex quantity z
- isolating δX then yields:

$$\delta X = \frac{-X^2 \Re\left\{ \left\langle \delta\vec{\xi}, \vec{\xi} \right\rangle \right\} + \Re\left\{ \left\langle \delta\vec{\xi}, \mathcal{F}(\vec{\xi}) \right\rangle \right\}}{X \left\langle \vec{\xi}, \vec{\xi} \right\rangle} = \frac{\Re \left\langle \delta\vec{\xi}, -X^2\vec{\xi} + \mathcal{F}(\vec{\xi}) \right\rangle}{X \left\langle \vec{\xi}, \vec{\xi} \right\rangle}$$

Symmetry of the pulsation equations

Consequence 3 – variational principle

$$\delta X = \frac{\Re\left\langle \delta\vec{\xi}, -X^2\vec{\xi} + \mathcal{F}(\vec{\xi})\right\rangle}{X\left\langle \vec{\xi}, \vec{\xi}\right\rangle}$$

- if $-X^2\vec{\xi} + \mathcal{F}(\vec{\xi}) = \vec{0}$, then $\delta X = 0$ for *any* $\delta\vec{\xi}$
  - however, $-X^2\vec{\xi} + \mathcal{F}(\vec{\xi}) = \vec{0}$ implies that $\vec{\xi}$ is an eigenmode, and X^2 its eigenvalue
- hence, this leads to the variational principle:

  A small error on an eigenfunction, $\delta\vec{\xi}$, leads to a 2^{nd} (or higher) order modification of X